# Introducing Free Choice

## Competition and Vouchers in Markets Supplied by Public Providers

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**Abstract**

This paper studies a classic economic question: What is the impact of introducing private competition partially financed by vouchers on quality, prices, sorting, and welfare in a market previously supplied by public providers only? It does so in a quantity-then-price oligopoly game à-la-Perlof-Salop with horizontal and vertical differentiation. Public providers' objective function is a weighted mean between profits and market share, whereas private providers' objective is profits. Introducing competition and vouchers in markets with horizontal and vertical differentiation has ambiguous effects on prices, quality, sorting, and consumer welfare since horizontal differentiation breaks the positive relation between quality and prices that emerges when there is only vertical differentiation. Sufficient conditions are provided for competition and vouchers to increase consumer welfare.

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# 1 Introduction

Governments face increasing pressure to provide high-quality public services at reasonable costs. This pressure has led governments to open several sectors to private providers' competition and to provide customers with vouchers, driven by the belief that competition results in high-quality services at low prices.[1] As a result of this trend, competition between public and private providers is now ubiquitous in healthcare, education, transportation, waste management and collection, and prison and security services. However, in many countries, this policy is controversial since people perceive that private providers prioritize profits at the expense of quality.

In the context of education markets, **?** argues that the solution to the low-quality problem is the introduction of competition through private schools,[2] partially financed by a generalized voucher, which will allow families to choose for-profit private schools thereby making the school system more competitive and more equitable since disadvantaged students could access better schools, competition induces investment in quality and improves the matching between schools and families.[3] The underlying assumptions behind his argument are that families are heterogeneous and free to choose, and the decision made by any family is independent of that made by any other family; i.e., families do not have preferences for peer groups.[4] From a policy perspective, defenders of free choice argue that it is "a tide that lifts all boats." Friedman's argument is extendable to any sector where public providers cannot accommodate heterogeneity in preferences, offer poor-quality goods and services, and face difficulties adapting to changing conditions.

This paper studies the introduction of for-profit private providers partially financed with vouchers in markets previously supplied exclusively by public providers to evaluate the claim that introducing private competition and vouchers improves quality with-

[1]See, for instance, **?** for data on expenditures on general government outsourcing as a percentage of GDP for OECD countries in 2016. The average share of spending outsourced across countries was 25%.

[2]The idea of vouchers has a long history; Tom Paine proposed a voucher in 1792 in his book "The Rights of Man."

[3]**??** never explicitly mention quality or that competition will improve the quality of public schools. The sentence "the development and improvement of all schools" leads some to interpret it as an improvement in the quality of public schools. Friedman's primary argument is that private schools improve the matching between family preferences and providers.

[4]See, also, **?** and **?**.

out leading to significant price increases and offsetting any adverse effects on public providers' quality resulting from more excellent sorting or cream-skimming. More specifically, we ask the following questions: i) When does privately-provided quality increase with the voucher? ii) Under what conditions does the introduction of private providers and vouchers increase the quality of public providers?; iii) Under what conditions privately-provided quality exceeds publicly-provided quality?; iv) Under what conditions do vouchers skim the cream; and v) When does introducing private providers increase consumers' welfare?

To answer them, we propose a model that combines the classical vertical differentiation model of quality-price competition by **??** and **??** with the horizontal differentiation oligopoly model **?**'s (**?**), in which providers first determine the quality of their goods/services and then set prices. By employing a well-established model, we aim to understand how the classical strategic effects–the business stealing and the strategic commitment effect of quality—-are affected by the introduction of competition and vouchers. Undoubtedly, models considering another dimension, such as incomplete information, search costs, peer preferences, etc., could be regarded as, but understanding the classic strategic trade-offs present in standard oligopoly games where providers choose qualities before prices are crucial to evaluate the claim that free choice is a tide that lifts all boats.

The paper considers a unit-demand model with more than one exogenously given private provider and more than one public provider. Public providers' objective function is a convex combination of their market share and profits, whereas private providers maximize profits. Quality not only has a direct cost but also increases the marginal cost of serving customers. Each customer is entitled to a non-discriminatory voucher paid when patronizing a private provider and a non-discriminatory subsidy paid when patronizing a public provider.

Customers' preferences are linear in the two dimensions of differentiation: quality valuation and nonpecuniary preferences shocks. The marginal utility of quality is independent of income and social interactions, such as the effects of peers. These are neither built into customers' preferences nor the production technology. Different customers have different marginal utility of quality drawn independently from the same distribution. Non-pecuniary preference shocks are i.i.d. across providers and customers and distributed log concave with compact and finite support.

The timing of the game is as follows: In the first stage, public and private providers choose quality levels simultaneously. After firms and customers observe qualities, private and public providers simultaneously select prices. Then, customers observe their random non-pecuniary preference shocks and the marginal utility of quality and choose providers.

Our setting is well-suited for markets where the provision of goods and services is undertaken by a small number of competitors, such as education, health, security, and transportation because the model considers quality and price strategic responses of both private and public providers when providers can more easily adjust prices than quality levels, as the latter are chosen before prices. The setting is also appropriate for studying choice and competition, as customers are free to choose between private and public providers, exhibit heterogeneous preferences, and the goods are vertically and horizontally differentiated, capturing a common feature in markets where free choice is adopted.

There is a unique equilibrium in the pricing sub-game for any quality profile. In this, prices are non-increasing in both the voucher and the per-customer subsidy. These results are due to demand being log-concave in its price and log-supermodular in prices since demands depend on price differences. Therefore, the price elasticity of demand falls with its price and rises with competitors' prices. However, prices may either rise or fall with quality. Because an increase in quality increases the marginal cost of production and the price elasticity of demand, the corresponding firm's price best response rises with it. However, a quality increase, holding prices constant, may decrease competitors' price elasticity of demand, lowering their best responses. Due to their strategic complementarity, the former raises prices and lowers them. Thus, a priori, higher quality may result in higher or lower prices. We provide sufficient conditions for prices to increase in their corresponding quality.

The main reason is that when there is vertical and horizontal differentiation, customers' utility ranking of providers is not the same for everyone, even though everyone prefers higher quality. Because of this, the positive equilibrium relationship between prices and quality that results when providers are vertically differentiated breaks, making the impact of higher quality on prices more nuanced.

The equilibrium quality profile results from the trade-off between the following forces: Firstly, the business-stealing effect that corresponds to the new customers drawn to the provider due to the higher quality when competitors' prices are held constant. The new customers come from both competing private and public providers.

Secondly, the strategic-commitment effect measures the customers lost or gained due to the induced change in competing providers' equilibrium prices with the provider's quality. When a competitor raises its price in response to the increase in quality, the strategic-commitment effect corresponding to that competitor implies a gain in customers since they would be paying higher prices if they kept their choice of provider unchanged. In contrast, when a competitor's price falls, the firm loses customers because the competitor becomes more attractive.

The customers drawn to the quality-increasing provider are not randomly drawn from competing providers–they are the ones that value quality the most–. In contrast, those who stay with competing providers are the ones with lower valuation for the increase in quality. Hence, an increase in quality results in a selection effect that raises customers' willingness to pay for the good.

Thirdly, there is a cost effect corresponding to the increase in the marginal cost of serving a customer with higher quality times the number of customers plus the direct cost of improving quality. Thus, the larger the number of customers patronizing a provider, the higher the total costs and the lower the average costs of providing quality.

The behavior of the equilibrium quality profile concerning the introduction of private competition and the voucher is challenging to characterize since the impact of the voucher on the forces determining quality is ambiguous. Price-cost margins may either rise or fall with the voucher since the pass-through from the voucher to the price could be either larger or smaller than the increase in the voucher. These features make it difficult to derive the impact of vouchers on quality.[5] Thus, to study the relationship between vouchers and

[5]**?** provides sufficient conditions for monotonicity in two-stage games. The conditions are demanding,

quality in the mixed market equilibrium, we focus on symmetric-by-sector equilibrium and impose conditions on equilibrium outcomes.

We provide sufficient conditions for the following results to occur: Firstly, privately-provided quality is larger than publicly-provided quality when the voucher is more significant than or equal to a threshold. Secondly, public providers' equilibrium quality in a mixed market exceeds that in a market supplied only by public providers when the voucher is smaller than a given threshold. Thirdly, private competition and vouchers result in cream skimming; i.e., high-quality valuations customers choose private providers more often than public providers when the voucher exceeds a given threshold. Fourthly, consumer welfare rises in a mixed market relative to a market where everyone is served by a public provider when the voucher exceeds a given threshold.

These results assume that an increase in the voucher increases the difference between the indirect utility of patronizing private providers and that of patronizing public providers. Because of horizontal differentiation, some customers choose public providers even though vouchers may lower the indirect utility of patronizing public providers.

Three assumptions are crucial for these results to hold. The first two are similar to the standard dominant-diagonal condition imposed in many oligopoly games to derive unambiguous comparative statics. Firstly, the Hessian for the price choices is a $B_0$-matrix in $p$ for any given quality profile, which implies that the price difference between the private and public sectors is decreasing with the voucher. Secondly, the Hessian for the quality choices is a $B_0$-matrix in $q$, and private providers' marginal profit from quality increases with the voucher more than that for public providers. This makes the difference between private and public providers' quality increase in the voucher. These two assumptions ensure that the difference in indirect utility between private and public providers raises with the voucher. Thirdly, the distribution of customers' best alternatives is convex (See, for instance, **?**). This assumption makes competition more intense and induces firms to act more aggressively when choosing qualities. We show that this holds whenever the number of competitors is sufficiently large, irrespective of the preference shock distribution.

difficult to check in non-linear oligopolies, and not easily satisfied in most markets.

Our results highlight a more nuanced picture of the consequences of introducing for-profit private competition and a generalized voucher in a market previously supplied by public providers only than suggested by **?** and his followers. Namely, the sufficient conditions for free choice to result in higher quality, better matches, and larger customer welfare are very stringent. The difficulty happens because when heterogeneous customers choose between public and private providers, they trade off quality and price differences at different rates due to vertical and horizontal differentiation. As a result of this, the strategic-commitment and business stealing effect of quality are non-monotonic in the voucher. In general, monotonicity requires quite specific behavior in terms of both the price and quality elasticity of demand with the voucher. The only reasonable policy implication that emerges from the analysis is that policymakers must analyze the market very carefully before introducing private competition partially financed with vouchers in markets previously supplied by public providers when the goal is to increase quality and not to raise coverage.

The rest of the paper is as follows: in the next Section, we discuss the related theoretical literature. In Section 3, the model and main assumptions are presented. In the next Section, we derive the equilibrium in the pricing sub-game. In Section 5, we derive the symmetric-by-sector sub-game-perfect equilibria, study some comparative statics, and consumer welfare. Section 6 briefly discusses the evidence on education and health markets where competition between private and public providers with vouchers is pervasive. Finally, in Section 7, we present concluding remarks.

## 2 Related Literature

This paper speaks to three different types of literature: The literature on product differentiation, the one in mixed oligopolies, which studies competition between private and public providers,[6] and the one on vouchers and free choice.

The first one is well-known, so we will not discuss it here. The second, while it speaks to the competition between public and private providers, focuses on how the presence

[6]See. for instance, **?**, **?**, **?**, and **?**.

of a public provider may improve welfare. In contrast, we focus on how competition and vouchers affect prices, qualities, and welfare. In addition, we consider multiple providers, vertical and horizontal differentiation, and vouchers simultaneously.[7] Fundamental features if one wishes to analyze the impact of competition between private and public providers on quality. The third one is the closest to this paper, despite its focus only on education markets.

**?** study a competitive market with free entry of private and public providers, perfect price discrimination, and peer effects. They show that equilibrium provision of education by public and private schools has a *cream skimming* effect on the wealthiest and most-able students, and universal vouchers lead to a further skimming effect.[8] **?** shows that vouchers that condition on students' ability fail to significantly affect cream skimming in the absence of further restrictions on schools' tuition and admission policies. However, when co-payments are limited, targeted vouchers achieve the dual goal of increasing quality and avoiding social stratification. Schools must participate in the voucher system for this to hold, and taxes might not increase.

**???** study the effects of voucher programs in computational general equilibrium models with multi-district local economies where there are public and private providers. He studies the impact of several voucher programs under alternative public school financing schemes using US Data.[9] These papers find that student sorting, the distribution of high-achieving peers, and the incentives for quality change when schools compete where parents have peer preferences. In addition, vouchers tend to exacerbate social stratification across different dimensions, and targeting helps to ameliorate this problem. Geographical mobility helps to deal with the perverse incentives to sort based on social attributes that competition creates.

[7] **?** study a quality-then-price game between a public provider and a private provider when there is vertical differentiation. They show the existence of multiple equilibria: one in which the public provider chooses lower quality and one in which the private provider chooses higher quality. Multiplicity is due to qualities being strategic substitutes and quality valuations being heterogeneous.

[8] **?** solves the existence of equilibrium problem in **?** by extending **?** using considering the possibility that households randomize among schools.

[9] See, **?** who extends Nechyba's model also to consider the possibility of religious private schools and **?** solves the existence of equilibrium problem in **?** by extending **?** by considering the possibility that households randomize among schools.

**?** is more closely related to this paper. He considers how the quality of a public school responds to the introduction of private schools and vouchers when there is free entry into the private sector. He considers identical private schools, co-payments are not allowed, homogeneous preferences, low- and high-income families, and higher quality means higher marginal costs of serving customers but not a direct cost. The free-entry condition fully pins down the quality of private schools. He shows that increasing the voucher may lower public quality since vouchers can make the private sector more attractive for high-income families, thereby increasing the effort costs required to keep these students in the public sector. **?** study how vouchers affect households of different income levels in a setting where public schools maximize profits vis-a-vis one in which they act competitively. Simulations show that poor families in either case may not be better off with a generalized voucher. He concludes vouchers do not promote equal educational opportunities. These papers share the future that the pass-through from the voucher to customers is -1, which is true in our case only when there are only private providers. The market is fully covered, assume away horizontal differentiation between providers, and consider homogeneous quality valuations.

Qualities in mixed provisions are often discussed in the education and health sectors. However, perspectives such as political economy, taxation, and income redistribution are incorporated, so public providers are typically assumed to have objective functions different from social welfare. **?** combines customers' voting and quality choices by public and private schools and lets the public provider be a Stackelberg leader. They show that when the government sets high quality, private schools choose a low one and vice-versa. This is similar to **?**, who study a quality-then-price game between a public and a private provider and vertical differentiation. The public firm maximizes social surplus, whereas the private firm maximizes profit. They show there are multiple equilibria. In some, the public firm chooses a low quality, and the private firm chooses a high quality. In others, the opposite is true.

None of these papers deals with strategic private and public providers that choose qualities first and then prices in a vertical and horizontal differentiation setting. Key dimensions in markets where private and public providers compete and quality is a key strategic variable. Technically, both types of differentiation break the positive relationship between prices and qualities that arises in models where only vertical differentiation is considered.

Another critical difference is that quality results from the selection of families across schools, and it is not the outcome of the investments made by schools as in this paper. Because in most papers, the quality in each school is entirely determined by the mean ability of the students that sort into the corresponding school, many of the incentives to invest in quality that competition between private and public schools and vouchers create are muted. Vouchers improve the selection of students in private schools and worsen that in public schools, and, as a result, quality in the public sector decreases. Finally, most papers in the literature assume a passive public sector or a non-strategic behavior by the public sector. This differs from the mechanism underlying our results since we do not consider conventional peer effects. Hence, the reason for the selection of customers across industries is the combination of preference heterogeneity with price and quality differences between the private and public sectors, and it is not due to income differences or peer effect preferences as is the case in most papers.

# 3 The Environment

We consider a market for goods or services with three different types of agents: customers, each consuming one unit of the good; the private sector; and the public sector, denoted by superscript 0.

**Providers** There are $n$ private providers indexed by $j \in \mathcal{J} \equiv \{1, \ldots, n\}$ and $N$ public providers indexed by $j \in \mathcal{J}^0 \equiv \{n+1, \ldots, n+N\}$. When a customer patronizes a private provider, this gets a voucher $v \in \Re_+$; when it patronizes a public provider, it gets a per-customer subsidy $g \in \Re_+$. Furthermore, each public provider receives a fixed transfer $T \geq 0$.

Provider $j$'s total cost of serving $s^j$ customers when its quality is $q^j$ is $\mathcal{C}^j(s^j, q^j) =$

$c^j(q^j)s^j + C^j(q^j)$.[10] This means that for a given quality level $q^j$, the marginal cost of serving a customer is constant and equals to $c^j(q^j)$, where $c^j(\cdot)$ is non-negative, strictly increasing and convex, with $c^j(\underline{q}) \geq 0$, $c^j_{q^j}(\underline{q}) = 0$ and $\lim_{q^j \to \infty} c^j(q^j) \to \infty$. For a given quality level, there is also a fixed cost of production which is $C^j(q^j)$, where $C^j(\cdot)$ is non-negative, strictly increasing and convex in $q^j$, with $C^j(\underline{q}) \geq 0$, $C^j_{q^j}(\underline{q}) = 0$ and $\lim_{q^j \to \infty} C^j(q^j) \to \infty$. Thus, for a given quality level, the average cost decreases with the number of customers, and the production technology exhibits economies of scale. We can think of quality as requiring investments in fixed inputs like capital goods and variable inputs like labor or more skillful labor.

Because we remain agnostic about whether producing goods and services is more or less expensive in the private sector than in the public sector, we have assumed that the total cost of serving any given number of customers at any given quality level by a public provider is the same as that by a private provider. When the cost of labor and capital determine the cost of quality and agency problems are equally severe in the private and public sector, this is the proper assumption.

Private providers aim to maximize profits, while public providers aim to maximize a weighted mean of profits and the demand (market share) they capture as in **?**. They place a weight $\beta$ on profits and $1 - \beta$ on their demand. Thus, public providers are partially rent-seekers. This stacks the deck against finding that incentives have perverse effects, as we consider the case in which incentives might be expected to be needed. Suppose they chose instead to maximize quality or only market share. In that case, it is unlikely that incentives would have any efficiency effects when there is competition between private and public providers. We suppose public providers earn rent by charging positive prices when needed. Because public providers stand to lose funding when the number of served customers falls, on the margin, their incentive is to retain customers in the face of private providers' competition. It is also common to see that public providers are rewarded not by their performance but by their enrollment or the share with the corresponding population that they serve.

Public providers could hold many other plausible objective functions such as a weighted

[10]This implies that this is a endogenous sunk-cost model.

average between profits and costumers' welfare, industry-wide quality, understood as the sum of each private and public provider's quality weighed by its corresponding demand, or public quality weighed by the public sector demand. We have chosen maximization of the weighted average between profits and demand because we wish to give the best opportunity to the argument that choice and vouchers increase competition among private and public providers, and the latter responds by increasing quality.

**Customers** Customers have unit demands, and the value of their outside option is normalized to zero. We model horizontal product differentiation by adopting a random-utility framework in the spirit of **?**. The utility of a customer when patronizing a provider that sets a price $p$ and offers quality $q$ is given by: $U(y,q,p,\theta)+\epsilon$, where $U(y,q,p,\theta) \equiv y+\theta q-p$, $\theta$ is the marginal valuation for quality, and $\epsilon$ is a non-pecuniary random utility shock that is specific to each provider. We assume that $\epsilon^j$ is i.i.d. across individuals, which reflects idiosyncratic tastes for different firms, and for a given customer, it is also i.i.d. across firms. $\epsilon^j$ is distributed $G(\cdot)$ with compact and full support $[\underline{\epsilon},\bar{\epsilon}] \subset \Re$ and zero mean. In addition, $g(\cdot)$ is twice continuously differentiable everywhere, is bounded, and log-concave and $g'(\cdot)$ is bounded. Quality valuation $\theta$ is distributed with cumulative distribution function $F(\theta)$ with full support $\Theta \equiv [\theta_L,\theta_H]$, density $f(\theta)$ , and mean $\bar{\theta}$. In addition, $f(\theta)$ is twice continuously differentiable everywhere and log-concave. Income is distributed $H(\cdot)$ with compact and full support $[\underline{y},\bar{y}] \subset \Re$, mean $y_m$. In addition, $h(\cdot)$ is twice continuously differentiable everywhere and log-concave.

The functional form of $U(y,q,p,\theta)$ for preferences assumes the following: first, as in **?**, utility is linear in quality, and quality and customers' valuations are complements. Thus, each customer has a different marginal valuation of quality; second, as in **?**, the utility function is additive in the two dimensions of differentiation. These two things imply that increases in the quality level are valued equally by customers with the same valuation irrespective of their location. This is a standard assumption in the differentiation literature that allows the two dimensions of differentiation to be identified (**?**, **?** and **?**); third, conditional on the valuation level, utility exhibits constant marginal utility of quality; and fourth, the marginal utility of quality and income are independent.

**Timing** At stage 1, public and private providers choose their quality levels simultaneously. At Stage 2, after firms and customers observe the quality profile, public and private providers determine prices simultaneously. At the final stage, customers observe the realization of their non-pecuniary utility shock from patronizing the private and public sectors and the non-pecuniary utility shock from patronizing each firm, the quality and price profile, and choose a provider. All households buy good or services from one of the available providers.

# 4 The Pricing Sub-Game

## 4.1 Demand Characterization

Once a customer learns his random-utility shocks, he chooses the provider with the highest utility. Thus, the customer chooses provider $j \in \mathcal{J}$ whenever $U(y, q^j, p^j, \theta) + \epsilon^j \geq U(y, q^k, p^k, \theta) + \epsilon^k$ for all $k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}$. Hence, the demand for provider $j$ is given by

$$D^j(p,q) = \mathbb{P}[U(y, q^j, p^j, \theta) + \epsilon^j \geq \max_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} \{U(y, q^k, p^k, \theta) + \epsilon^k]$$

It readily follows from this that

$$D^j(p,q) = \mathbb{E}_{y,\theta,\epsilon}\Big[\prod_{k \in J \cup \mathcal{J}^0 \setminus \{j\}} G^k\Big(U(y, q^j, p^j, \theta) + \epsilon^j - U(y, q^k, p^k, \theta)\Big)\Big]$$

where the equality follows from the independence assumption about the $G$s distributions and where $q \equiv (q^1, \ldots, q^n, q^{n+1}, \ldots, q^{n+N})$ and $p \equiv (p^1, \ldots, p^n, p^{n+1}, \ldots, p^{n+N})$.

**Proposition 1.** *For any* $(p,q) \in \Re_+^{2(n+N)}$*,*

*i)* $D^j(p,q)$ *is decreasing and log-concave in* $p^j$*, increasing in* $p_{-j}$*, and log-supermodular in* $p$*.*

*ii)* $D^j(p,q)$ *is increasing and log-concave in* $q^j$*, decreasing in* $q^{-j}$*, log-submodular in* $q$*, and log-supermodular in* $(p^j, q^j)$*.*

The log-concavity of the provider-specific demand implies that the price elasticity of demand decreases with its price. The log-supermodularity in $p$ means that the price elasticity of demand decreases as competitors' prices increase. The latter will imply an increasing best-response correspondence when marginal costs are constant or convex and goods are gross substitutes.

## 4.2 Equilibrium Prices

Let $\Pi^j(p,q;\beta^j) \equiv \big(\beta^j(p^j + v - c^j(q^j)) + 1 - \beta^j\big)D^j(p,q)$, where $\beta^j = 0$ for all $j \in \mathcal{J}$; i.e., when provider $j$ is private, and $\beta^j = \beta^j$ for all $j \in \mathcal{J}^0$; i.e., when provider $j$ is public. Provider $j$'s goal is to maximize $\Pi^j(p,q;\beta^j) - C^j(q^j)$ with respect to $p^j$, but since $C^j(q^j)$ is independent of $p^j$, it faces the following monotonically transformed optimization problem

$$\max_{p^j \in \Re_+} \log \Pi^j(p,q;\beta^j). \tag{1}$$

Provider $j$'s first-order condition, when the price is positive, is given by

$$\frac{p^j(q) + g - c^j(q^j)}{p^j(q)} = -\frac{1}{\eta^j(p(q),q)} - \frac{1-\beta^j}{\beta^j}\frac{1}{p^j(q)}, \tag{2}$$

where $\eta_j(\cdot)$ is the price elasticity of demand given by

$$\eta^j(p,q) = -\frac{p^j D^j_j(p,q)}{D^j(p,q)}$$

and

$$D^j_j(p,q) = -\mathbb{E}_{y,\theta,\epsilon}\Big[\sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(U(y,q^j,p^j,\theta) + \epsilon^j - U(y,q^k,p^k,\theta)\Big)\times \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(U(y,q^j,p^j,\theta) + \epsilon^j - U(y,q^k,p^k,\theta)\Big)\Big],$$

where $\nu_g(\cdot) \equiv g(\cdot)/G(\cdot)$.

The first term in $\eta_j(p,q)$ is the decrease in demand due to customers switching to patronize other private or public providers. The customers switching to other providers are not randomly selected among the providers patronizing provider $j$. Customers with higher quality valuations stay with provider $j$ when their quality is higher than or equal to competitors' providers. In contrast, those with low valuations stay with provider $j$ when the opposite occurs. The main difference between a public and a private provider is that the former, ceteris paribus, is less concerned with profit margins. Thus, it chooses a lower price than an identical private provider that offers the same quality when each faces the same number of competitors or each type.

**Proposition 2.** *For any $(q,v,g) \in R_+^{n+N+2}$, there is a unique pure strategy Nash equilibrium in the pricing sub-game.*

Existence and uniqueness follow from log-concavity, log-supermodularity, constant marginal costs, and the fact that any provider's demand depends only on price differences and not price levels.[11] Because firms' best responses are increasing since demands are log-supermodular in $p$, there is the lowest and highest equilibrium. Because demands are log-concave in their price, i.e., the price elasticity of demand falls with the price, the slope of the best response for each firm is lower than one, and therefore, there is a unique fixed point.

Since the transformed game is supermodular in $p$ and has a unique equilibrium, it follows from Theorem 5 in **?** that each player has only one serially undominated strategy. Because the set of serially undominated strategies is determined only by ordinal comparisons, the corresponding prices are also the unique serially undominated strategies in the original game. Hence, the original game has a unique equilibrium. This is dominance solvable and globally stable under any adaptive learning rule satisfying assumption A6 in **?**.

[11]This result holds when costs are convex.

## 4.3 Comparative Statics

The next result provides comparative statics concerning $(q, g, v)$. This readily follows from the fact that the game is log-supermodular in $(-g, -v)$ and private providers' best responses fall with the voucher since $D^j_j < 0$ and that for public providers is independent of the voucher.

**Proposition 3.** *For any $(q, v, g) \in R^{n+N+2}_+$, the equilibrium price profile $p(q)$ is non-increasing with $(g, v)$.*

This proposition shows that equilibrium prices, holding everything else constant, fall with the voucher since the higher the voucher and the public sector subsidy. Because an increase in the voucher raises the private providers' price-cost margins, which makes a higher price less profitable, public providers' best responses remain unaltered, and prices are strategic complements, similarly, for an increase in the per-customer public sector subsidy.

The effect of an increase in provider $j$'s quality on provider $j$'s equilibrium price is ambiguous since payoffs are not supermodular in $(p, q)$. Namely, firm $j$'s best response increases with $q_h$ whenever

$$\Pi^j_{jq^h}(p, q; \beta^j) = \frac{1}{(p^j(q))^2}\left(c^j_{q^h}(q^j)(\eta^j(p(q), q))^2 + p^j(q)\frac{\partial \eta^j(p(q), q)}{\partial q^h}\right) > 0, \tag{3}$$

where

$$D^j_{q^j}(q)\big|_{p^j=l} = \mathbb{E}_{y,\theta,\epsilon}\Bigg[\theta \sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(U(y, q^j, p^j, \theta) + \epsilon^j - U(y, q^k, p^k, \theta)\Big)\times$$
$$\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(U(y, q^j, p^j, \theta) + \epsilon^j - U(y, q^k, p^k, \theta)\Big)\Bigg].$$

On the one hand, an increase in $q^j$ increases the marginal cost of production, which, ceteris paribus, raises firm $j$'s best response and therefore raises prices since they are strategic complements. On the other hand, it changes the price elasticity of demand. Holding prices constant, firm $j$'s price elasticity of demand increases since $D^j_{jq^j}D^j - D^j_j D^j_{q^j} > 0$

and therefore firm $j$'s price best response rises with $q^j$. Thus, firm $j$'s price best response raises with $q^j$, which pushes prices up due to strategic complementarity. However, an increase in $q^j$, holding prices constant, decreases competitors' price elasticity of demand whenever $D^k_{kq^j}D^k - D^k_k D^k_{q^j} < 0$. Because of this, firm $k$'s price best response falls with $q^j$ since $q^j$ does not affect firm $k$'s marginal costs. This force pushes prices down due to strategic complementarity. Thus, prices may rise or fall with an increase in $q^j$.

A consequence of adding horizontal differentiation to a model of vertical differentiation with linear preferences is to break the positive relationship between prices and qualities. When there is vertical and horizontal differentiation the ranking of providers in terms of utility –quality and price– is not the same for everyone despite everyone preferring higher quality.

To find a sufficient condition for $p^j(q)$ to increase with $q^j$, we use the Implicit Function theorem and the properties of $B_0$-matrices.[12] A $B_0$-matrix is one in which the mean of each row is positive and greater than the maximum between zero and each off-diagonal element in the same row. For instance, this is true for a diagonally dominant matrix. $B_0$ matrices have strictly positive diagonal and positive determinants, and their principal sub-matrices are all $B_0$-matrices, which means positive determinants too. Another valuable property is that the sum of co-factors is positive for each row. Certainly, the same conclusions apply if the transpose of a matrix is a $B_0$-matrix since the determinant of a matrix equals the determinant of its transpose.

Let $H(p, q; \beta)$ be the Hessian with respect to $p$ with entries $\left(\frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial p^h}\right)_{h,j \in \mathcal{J} \cup \mathcal{J}^0}$ and $H_q(p, q; \beta)$ be the matrix with entries $\left(\frac{\partial^2 \log \Pi^j(p,q;\beta^j}{\partial p^j \partial q^h}\right)_{h,j \in \mathcal{J} \cup \mathcal{J}^0}$. The matrix $H$ is a $B_0$-matrix when for all $j \in \mathcal{J} \cup \mathcal{J}^0$,

$$\sum_{h \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi^j(p, q; \beta^j)}{\partial p^j \partial p^h} \leq (n + N) \min_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} \left\{0, \frac{\partial^2 \log \Pi^j(p, q; \beta^j)}{\partial p^j \partial p^h}\right\}.$$

[12]See, **?** and **?**, who uses $B_0$ matrices to derive comparative statics.

and the matrix $-H^T$ is a $B_0$-matrix when for all $j \in \mathcal{J} \cup \mathcal{J}^0$,

$$\sum_{h \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi^h(p, q; \beta^j)}{\partial p^h \partial p^j} \leq (n+N) \min_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} \left\{0, \frac{\partial^2 \log \Pi^h(p, q; \beta^j)}{\partial p^h \partial p^j}\right\}.$$

Thus, the matrix $-H^T(p, q; \beta)$ is mean positive dominant in element $i$ for all $i \in \mathcal{J} \cup \mathcal{J}^0$.[13]

**Proposition 4.** *Suppose that $(q, v, g) \in R_+^{n+N+2}$ are such that $p(q) > 0$,[14] $-H^T(p, q; \beta)$ is a $B_0$-matrix, and $-H_q(p, q; \beta)$ is mean positive dominant in element $j$. Then, the equilibrium price $p^j(q)$ is non-decreasing in $q^j$ and $p^h(q)$ may either rise or fall with $q^j$.*

Because of quasi-linear consumer preferences, a price reduction would have the same effect on demand as a quality increase if $\theta$ were to be a fixed parameter since $D^j(p^j, p^{-j}; q^j + \delta, q^{-j}) = D^j(p^j - \theta\delta, p^{-j}; q)$. However, because different customers have different valuations, an increase in quality induces switching from customers with higher quality valuations. This makes the comparative statics concerning $q$ different from the one that emerges from an exogenous decrease in the price despite the utility differences depend on quality and price differences. It follows from this and the increasing price elasticity of demand with its quality and falling with competitors' quality that an unambiguous comparative statics demand to restrict the size of the indirect effects relative to taht for the direct effects.

The $B_0$-matrix assumption implies that the direct effect of an increase in $q^j$ on firm $j$'s price elasticity of demand is larger than the effect on firm $j$'s competitors' price elasticity of demand. Thus, the increase in firm $j$'s best response more than compensates for the decrease in competitors' best responses. When firms are symmetric and offer the same quality, this holds. However, when firms are either asymmetric or symmetric and provide different quality levels, this might not hold since the larger the difference between qualities, the larger the difference in the impact of private provider $j$'s quality on its price elasticity of demand and the impact on competitors' price elasticity of demand. Thus, we cannot even be sure that higher quality providers set higher prices in equilibrium. In addition, public providers place a positive weight on demand, making public providers'

[13]This condition is less stringent than the standard dominant diagonal condition. For details, see **?**.

[14]For any vector $x$, the notation $x > 0$ means that each component is strictly positive.

sensitivity to their own and competitors' quality different from private providers' sensitivity.

## 4.4 Exclusive Markets and Symmetric Mixed Markets

### 4.4.1 Exclusive Market with Symmetric Firms

Let's assume that firms are symmetric: i.e., $c^j(\cdot) = c(\cdot),\ \forall\ j \in \mathcal{J} \cup \mathcal{J}^0$ and the market is supplied only by either private or public providers. It readily follows from the first-order condition in equation (2) for $q^j = q$ for all $j$ that the symmetric equilibrium price is given by

$$p = \max\left\{0, c(q) - s - \frac{1-\beta^j}{\beta^j} + \frac{1}{m}\underbrace{\frac{1}{\int_{\underline{\epsilon}}^{\bar{\epsilon}} g(\epsilon) dG(\epsilon)^{m-1}}}_{\text{demand slope}}\right\}. \tag{4}$$

where $(m, s) = (N, g)$ and $\beta^j = \beta$ when providers are public and $(m, s) = (n, v)$ and $\beta^j = 1$ when they are private.

The numerator in equation (4) is the equilibrium demand. The denominator is the slope of the demand. This term is the density of a firm's marginal customers –those indifferent between the corresponding firm and the best alternative firm for them times the loss from a lower probability of being patronized. It is easy to see that the pass-through from subsidies to prices is -1, and from quality to prices is $p'(q) = c'(q)$.

Let $q^0$ be the public sector quality and $q^1$ the private sector quality when firms are symmetric within a sector. Then, when $v = g$ and $q = q^0 = q^1$, it readily follows from equation (4) that the difference between private and public providers' price is $-\frac{1-\beta}{\beta}$. Thus, the equilibrium price in a purely publicly-provided market is lower than the equilibrium price in a strictly private market for $g \leq v - (1-\beta)/\beta$ and $q^0 \leq q^1$, and the difference raises with $\beta$ since a larger $\beta$ implies that public providers care relatively more about profits than market share. In the limit when $\beta$ goes to 1, the price charged by pubic providers is identical to that set by private providers when $v = g$, $n = N$, and $q^1 = q^0$, whereas when $\beta^j$ goes to 0, the price goes to zero.

### 4.4.2 Mixed Markets with Symmetric-by-Sector Firms

The case where all firms are symmetric within each sector is a natural benchmark that allows for further clear analytic results and intuition. In this section, we consider all firms to be identical- having the same marginal cost- and the same quality level within each sector. The quality profile is given by $q = (q^1, q^0)$ and the price profile $p = (p^1, p^0)$.

**Proposition 5.** *Suppose that $(q, v, g) \in R^4_+$ are such that $(p^0(q), p^1(q)) > 0$.*

*i) If $n = N$, $q^1 = q^0 = \hat{q}$, and $g = v - \frac{1-\beta}{\beta}$, then $p^1(\hat{q}, \hat{q}) = p^0(\hat{q}, \hat{q})$. Furthermore, if the price elasticity of demand is non-increasing in competitor's quality and $q^1 > q^0 = \hat{q}$, $p^1(q^1, \hat{q}) > p^0(q^1, \hat{q})$ for all $g \geq v - \frac{1-\beta}{\beta}$.*

*ii) Suppose that $-H(p, q; \beta)$ is a $B_0$-matrix in $p$, then $p^1_v \leq p^0_v$.*

*iii) Suppose that $-H^T(p, q; \beta)$ is a $B_0$-matrix in $p$, then $p^1_{q^1} \geq 0$ whenever $\Pi^1_{1q^1}(p(q), q) \geq \max\{\Pi^0_{0q^1}(p(q), q), -(N/n)\Pi^0_{0q^1}(p(q), q)\}$ and $p^0_{q^1} \leq 0$ whenever $n\Pi^1_{1q^1}\Pi^0_{01} \leq \Pi^0_{0q^1}(\Pi^1_{11} + (n-1)\Pi^1_{1n})$.*

*iv) Suppose that $-H^T(p, q; \beta)$ is a $B_0$-matrix in $p$, then $p^1_{q^1} \geq p^0_{q^1}$ whenever $\Pi^1_{1q^1}(p(q), q) \geq 0 \geq \Pi^0_{0q^0}(p(q), q)$.*

The first part says that if qualities and competition intensity across sectors are the same, then prices are the same when the voucher is lower than the per-student subsidy in an amount $(1 - \beta)/\beta$ because of public providers' mandate to be concerned not only with profits but also with the share of the population they serve. Because of this public providers choose lower prices to make their sector more attractive. Furthermore, if private providers' quality is more significant, private providers' prices are larger than public providers'. When the voucher is equal to or smaller than the per-student subsidy minus $(1 - \beta)/\beta$ since prices fall with the voucher.

This follows from the fact that profits are log-concave in their own price, log-supermodular in competitors' prices, and firms' best responses increase with their own quality and decrease with competitors' quality. Thus, whenever private providers' quality rises, they increase prices, and public providers either increase them by a lower amount or decrease them.

The second part establishes an interesting result: if firms are symmetric within a sector and the Hessian satisfies the regularity condition posed in the proposition, then the pass-through from the voucher to private providers is larger than to public providers. Thus, holding quality constant, a large voucher decreases prices and makes private providers less expensive than public providers. A similar result is obtained (part (iv)) with respect to its own quality when the corresponding provider's best response rises with its quality and the competitor's best response does not increase with it. Thus, in this case, we recover, albeit in relative terms, that as a firm increases its quality, its price increases relatively more than competitors'.

The third part establishes conditions for prices to increase with its quality and decrease with competitors' quality. The increase in it quality requires that their price responses increase with $q^1$ and this increase is significant to compensate any decrease in public-providers' best response. Public providers quality falls with private providers' quality when an increase in competitors' quality decreases public providers' best response by a significant amount relative to the increase in private providers' best responses adjusted by the ratio between the direct effect on private providers' profits and the degree of price complementarity for public providers.

In Table 1, we present comparative static regarding increases in public and private providers' quality when the equilibrium is symmetric within each sector. The first row is for $v = 6$ and the next considers $v = 7$.

The relationship between prices and vouchers is negative, as predicted. In every case, the pass-through from the voucher to private-provides prices is higher than -1, and it is even smaller for public providers' prices. Thus, $p_v^1 - p_v^0 < 0$.

The relationship between public providers' quality and equilibrium prices is monotonic. Prices increase in its quality and decrease in competitors' quality. The pass-trough from $q^1$ to prices is such that $p^1_{q^1} - p^0_{q^1} > 0$ and from $q^0$ is such that $p^1_{q^0} - p^0_{q^0} < 0$.

| $q^0$, $q^1$ | 16, 18 | 17, 18 | 18, 18 | 19, 18 | 20,18 | 18, 19 | 18, 20 |
|---|---|---|---|---|---|---|---|
| $p^0$, $p^1$ | 24.36, 33.65 | 28.24, 33.14 | 32.44, 32.44 | 36.82, 31.76 | 41.20, 31.25 | 31.76, 36.82 | 31.25, 41.20 |
| $p^0$, $p^1$ | 24.24, 32.71 | 28.12, 32.22 | 32.32, 31.55 | 36.72, 30.88 | 41.14, 30.35 | 31.65, 35.90 | 31.13, 40.26 |

**Table 1.** Comparative statics regarding prices concerning quality
. $v \in \{6, 7\}$, $g = 6 - (1-\beta)/\beta$, $c(q) = 0.1q^2$, $\beta = 0.8$, $n = N = 6$, $\epsilon \sim \mathcal{N}(0, 10; -30, 30)$ and $\theta \sim \mathcal{N}(10, 2; 1, 100)$.

# 5 The Quality Sub-Game

## 5.1 The Equilibrium

When we substitute the equilibrium price into the profit function, provider $j$'s profit maximization problem becomes:

$$\max_{q^j \in Q} \{\Pi^j(p(q), q; \beta^j) - C^j(q^j)\}.$$

Let's denote $D^j(p(q), q)$ by $D^j(q)$. Due to the envelope Theorem, provider $j$'s first-order condition for $q^j$ is as follows

$$(\beta^j(p^j(q) + s^j - c^j(q^j)) + 1 - \beta^j)D^j_{q^j}(q)|_{p^j=k} - \beta^j c^{j'}(q^j)D^j(q) - C^{j'}(q^j) = 0. \quad (5)$$

where $s^j \in \{v, g\}$ and the partial derivative of provider $j$'s demand concerning its quality when its price is held constant is equal to

$$\begin{aligned} D^j_{q^j}(q)\big|_{p^j=c} = \mathbb{E}_{y,\theta,\epsilon}\Bigg[\theta \sum_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} v_g\Big(U(y, q^j, p^j, \theta) + \epsilon^j - U(y, q^k, p^k, \theta)\Big) \times \\ \prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\Big(U(y, q^j, p^j, \theta) + \epsilon^j - U(y, q^k, p^k, \theta)\Big)\Bigg] + \\ \sum_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} D^j_h(p, q) p^h_{q^j}(q). \end{aligned}$$

The first-order condition for $q^j$ in equation (5) can be explained by three different effects: the income gained from the business-stealing effect, the income gained/loss from the strategic-commitment effect, and the loss due to a higher cost to provider higher quality.

The business-stealing effect corresponds to the new customers switching to provider $j$ due to the higher quality when competitors' prices are constant. They come from both competing private and public providers.

The strategic-commitment effect measures the customers lost or gained due to the induced change in competing providers' equilibrium prices with provider $j$'s quality. When a competitor raises its price as a response to the increase in quality, the strategic-commitment effect implies a gain in customers since they would be paying higher prices if they keep their choice of provider unchanged, whereas when a competitor's prices fall, firm $j$ loses customers because competitors become more attractive.

The customers drawn to provider $j$ by either effect are not randomly drawn from competing providers; they are the ones who value quality the most, whereas those who prefer competing providers have the lowest valuation for the increase in quality among all those who have not been choosing provider $j$ before quality improves.

Finally, the cost effect corresponds to the increase in the marginal cost of serving a customer with higher quality times the number of customers plus the direct cost of improving quality. The larger the number of customers patronizing provider $j$, the higher the total costs. Furthermore, the lower the $\beta^j$, the less provider $j$ cares about the marginal income minus marginal cost of quality and more about market share.

Private-provider $j$'s quality best response when the equilibrium conditions for prices are considered is the solution to the following equation

$$-\beta^j \frac{D^j(q)}{\eta^j(p(q),q)} \frac{p^j(q)}{q^j} \Bigg( \underbrace{\xi^j(p(q),q)}_{Bussiness\ Stealing\ Effect} + \underbrace{\sum_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} \eta^{jh}(p(q),q)\sigma^{hj}(q)}_{Strategic-Commitment\ Effect} \Bigg) + \qquad (6)$$

$$\underbrace{\beta^j \frac{D^j(q)}{\eta^j(p(q),q)} \frac{p^j(q)}{q^j} c^{j'}(q^j) - C^{j'}(q^j)}_{Cost\ Effect} = 0,$$

where $\xi^j(p(q),q) > 0$ is firm $j$'s quality elasticity of demand when prices are held constant, $\eta^{jh}(p(q),q) > 0$ is the cross-price elasticity of demand, and $\sigma^{hj}(q)$ is firm $j$'s quality elasticity of competitor $h$'s price. Thus, the first term inside the parenthesis measures the business-stealing effect and the second term is the strategic-commitment effect. Its sign is a-priori unknown since competitors' prices may rise or fall with $q^j$.

From now on, we assume the following:

**Assumption 1.** *For all $\beta^j \in [0,1]$, $\Pi^j(p(q), q; \beta^j) - C^j(q^j)$ is quasi-concave in $q^j$.*

A sufficient condition for this is that $D^j(q)$ is log-submodular in $(p^j, q^j)$ and $D^j_{q^j}(q)|_{p^j=k}$ falls with $q^j$.

The next result follows from the Debreu-Glicksberg-Fan's Theorem.

**Proposition 6.** *Suppose that Assumption 1 hold. Then, there exists a sub-game perfect equilibrium $(q(v,g), p(v,g)) \in \Re_+^{2(N+n)}$, with $q(v,g) > 0$ and $p(v,g) \geq 0$. If firms are symmetric, $\beta^j = 1$, $N = n$, and $v = g$ the equilibrium is symmetric, whereas if $\beta^j < 1$, there is no symmetric equilibrium.*

## 5.2 Quality and Vouchers When Markets Are Exclusive

In this sub-section, we compare the equilibrium when there are only symmetric private providers with that in which there are only symmetric public providers.

It readily follows from the first-order condition in equation (5) and the equilibrium condition for prices in equation (2) that the quality is the solution to[15]

$$\frac{\beta^j}{m}\bar{\theta} - C'(q) = 0, \tag{7}$$

where $(m,s) = (N,g)$ and $\beta^j = \beta$ when providers are public and $(m,s) = (n,v)$ and $\beta^j = 1$ when they are private.

It readily follows that the equilibrium quality, denoted by $q^1(s)$, is independent of $s$, raises with $\bar{\theta}$, falls with $m$, and raises with $\beta$.

The reason why the equilibrium quality is independent of the voucher is twofold: first, the pass-through from the voucher to prices is equal to -1, which implies that vouchers do

[15]The objective function is strictly concave in $q^j$ and, thereby, a sub-game equilibrium in which quality is positive exists and is unique.

not change profit margins; and second, the demand is independent of the voucher. These equilibrium features happen because customers' utility is linear on income (prices), the marginal utility of quality is independent of income, and there is full coverage.[16] Hence, the impact of prices on the marginal customer is independent of the quality. This is no longer the case when there are private and public providers.

Quality rises with $\beta^j$ since providers care more about profits and less about market shares, and prices increase with $\beta^j$, and therefore, the markup, holding quality constant, is higher.

Comparing the first-order conditions in equations (**??**) and (7), we deduce the following result.

**Proposition 7.** *Suppose within-sector-firms are symmetric and $(v, g) \in R^2_+$ is such that $p(q) > 0$. Then, privately-provided quality when customers are served exclusively by private providers is larger than publicly-provided quality when customers are served exclusively by public providers for all $\beta < 1$ whenever $n \geq N\beta$. Whenever $q^1 \geq q^0$, private providers' price exceeds public providers' price if $g \geq v - \frac{1-\beta}{\beta}$ and $n \leq N$.*

Because public providers focus on profits and market shares, they charge a lower price and offer lower quality. The quality they offer decreases as their weight on the market share $1 - \beta$ rises. This happens because public providers' prices rise with $\beta$, which means ceteris-paribus a larger profit margin, and higher quality yields a higher demand. Therefore, the marginal return to quality rises with $\beta$.

Under the full coverage assumption, neither vouchers nor per-customer subsidies impact quality. Any effect on quality results from competition between public and private providers and having different objective functions.

## 5.3 Comparative Statics: Quality and Vouchers

Let $H(q; \beta)$ be the Jacobian of the first-order conditions for qualities evaluated at $(p(v, g), q(v, g))$ (hereinafter the Hessian for short). By the Implicit Function theorem, we have that for all

[16] When coverage is not complete, the threshold for quality valuation that leaves a customer indifferent between patronizing a provider and not doing so will depend on the price and quality offered by the corresponding provider.

$j \in \{0,1\}$,

$$q_v^j(v,g) = -\frac{\sum_{i\in J\cup\mathcal{J}^0}\Pi^i_{q^i v}H^{ij}}{\sum_{i\in J\cup\mathcal{J}^0}\Pi^i_{q^i q^j}H^{ij}}\Big|_{(q(v,g),p(v,g))}. \tag{8}$$

where $H^{ij}$ is the $ij$ co-factor from the Hessian $H(q;\beta)$.

An increase in the voucher rises provider $j$'s best response when

$$\Pi^j_{q^j v} = \beta^j\left(\frac{D^j(q)D^j_{jv}(q) - D^j_j(q)D^j_v(q)}{(D^j_j(q))^2}D^j_{q^j}(q)|_{p^j=k} - \frac{D^j(q)}{D^j_j(q)}D^j_{q^j v}(q)|_{p^j=k} - c'(q^j)D^j_v(q)\right), \tag{9}$$

where

$$D^j_v(q) = \sum_{h\in\mathcal{J}\cup\mathcal{J}^0} D^j_h p^h_v(q) = \sum_{h\in\mathcal{J}\cup\mathcal{J}^0\setminus j} D^j_h(p^h_v(q) - p^j_v(q)),$$

$$D^j_{jv}(q) = \sum_{h\in\mathcal{J}\cup\mathcal{J}^0} D^j_{jh}p^h_v(q),$$

and

$$D^j_{q^j v}(q)|_{p^j=k} = \underbrace{\sum_{h\in\mathcal{J}\cup\mathcal{J}^0} D^j_{q^j h}p^h_v(q)}_{\text{Change in the Business}-\text{Stealing Effect}} + \underbrace{\sum_{h\in\mathcal{J}\cup\mathcal{J}^0\setminus j} D^j_h p^h_{q^j v}(q) + \sum_{i\in\mathcal{J}\cup\mathcal{J}^0}\sum_{h\in\mathcal{J}\cup\mathcal{J}^0\setminus j} D^j_{hi}p^h_{q^j}(q)p^i_v(q)}_{\text{Change in the Strategic}-\text{Commitment Effect}}.$$

The first term inside the parenthesis measures how the price-cost margin changes with the voucher when quality is constant. Because quantity demanded falls with its price and rises with competitors' prices and prices fall with vouchers, the equilibrium markup could either increase or decrease with the voucher. When the demand, evaluated at the equilibrium price profile $p(q)$ is log-convex in $(p^j, v)$, the term is positive, which means that the pass-through from the voucher to prices is larger than $-1$; that is, $p^j_v(q) \geq -1$.

The second term inside the parenthesis measures how the sum of the business-stealing and strategic-commitment effect changes with the voucher. The business-stealing effect raises with the voucher when the demand is submodular in $(q^j, p)$ since prices fall with the voucher. In contrast, the strategic-commitment effect does so when: (i) $p^j_{q^j}(q) \geq 0$ nd $p^h_{q^j}(q) \leq 0$ for all $h \neq j$ since firm $j$'s demand is log-concave in $p^j$ and supermodular in $p$ and (ii) $\sum_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} D^j_h p^h_{q^j v}(q) \geq 0$. Because $D^j_j = -\sum_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} D^j_h$, this holds whenever $\sum_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} D^j_h (p^h_{q^j v}(q) - p^j_{q^j v}(q)) \geq 0$. Because $D^j_h > 0$, this holds when the change in competitors' price response to provider $j$'s quality with a hike in the voucher is larger than the provider $j$'s price response with its quality.

The third term in equation (9) is negative since demand is decreasing. Therefore when demand is smaller, a higher marginal cost has a lower impact on total costs. To sign the strategic-commitment effect, we need to sign the change in each term of the Hessian for equilibrium prices. Doing so does not help us better grasp the intuition underlying the result.

**Proposition 8.** *Suppose $(v, g) \in R^2_+$ is such that $p(q) > 0$.*

*i) Suppose that $H(q; \beta)$ is a $B_0$-matrix in $q$.*[17] *If*

$$\min_{h \in \mathcal{J} \cup \mathcal{J}^0} \{\Pi^h_{q^h v}(p(q), q)\} \geq 0,$$

*then $\sum_{j \in \mathcal{J} \cup \mathcal{J}^0} q^j_v \geq 0$.*

*ii) Suppose that $H^T(q; \beta)$ is a $B_0$-matrix in $q$. If*

$$\sum_{h \in \mathcal{J} \cup \mathcal{J}^0} \Pi^h_{q^h v}(p(q), q; \beta) \geq (n + N) \max_{h \neq j} \{0, \Pi^h_{q^h v}(p(q), q; \beta^h)\},$$

*then $q^j_v \geq 0$.*

[17]Recall that this is less stringent than assuming dominance diagonal, the standard assumption in oligopoly comparative statics.

Aggregated quality increases with the voucher when the best responses regarding quality raise with the voucher, irrespective of whether qualities are complements or substitutes. When qualities are strategic complements, this is straightforward. In contrast, when they are substitutes, this requires that neither private nor public providers' best responses change by a large amount with the voucher since the drop in private providers' quality could be compensated by the increase in public providers' quality or vice-versa. The $B_0$-matrix assumption ensures that this is the case since it assumes that the average impact in marginal returns exceeds the most significant indirect impact.

The quality of firm $j$ raises with $q^j$ whenever firm $j$'s best response increases with $q^j$, and the average increase in best responses is larger than the largest increase in the best response of firm $j$'s competitors. This, together with the assumption that $H(q;\beta)$ is a $B_0$-matrix in $q$, ensures that the direct effect on firm $j$'s best response outweighs the indirect effects from competitors' optimal responses to a larger $q^j$. For this to hold, price responses to quality changes and how price-quality cross elasticities of demand respond to price changes must take a particular form.

**Corollary 1.** *Suppose within-sector-firms are symmetric and $(v, g) \in R^2_+$ is such that $p(q) > 0$.*

*i) Suppose that $H(q;\beta)$ is a $B_0$-matrix in $q$. If*

$$\min\{\Pi^1_{q^1 v}(p(q), q; \beta^j), \Pi^0_{q^0 v}(p(q), q)\} \geq 0,$$

*then $nq^1_v + Nq^0_v \geq 0$.*

*ii) Suppose that $H(q;\beta)$ is a $B_0$-matrix in $q$. Then, $q^1_v \geq 0$ whenever $\Pi^1_{q^1 v} \geq (\Pi^1_{q^0 q^0} + (N-1)\Pi^0_{q^0 q^N}) - N\Pi^0_{q^0 v}\Pi^1_{q^1 q^0} \leq 0$. A sufficient condition for this to hold is $\Pi^1_{q^1 v} > 0$ and $\operatorname{sign}\Pi^0_{q^0 v} = \operatorname{sign}\Pi^1_{q^1 q^0}$.*

*iii) Suppose that $H(q;\beta)$ is a $B_0$-matrix in $q$. Then, $q^0_v \geq 0$ whenever $n\Pi^1_{q^1 v}\Pi^0_{q^0 q^1} - \Pi^0_{q^0 v}(\Pi^1_{q^1 q^1} + (n-1)\Pi^1_{q^1 q^n}) \geq 0$. A sufficient condition for this to hold is $\Pi^0_{q^0 v} \geq 0$ and $\operatorname{sign}\Pi^1_{q^1 v} = \operatorname{sign}\Pi^0_{q^0 q^1}$.*

*iv) Suppose that $H(q;\beta)$ is a $B_0$-matrix in $q$. If either $\Pi^1_{q^1 v} \geq 0$ and $\Pi^0_{q^0 v} \leq 0$ or $\Pi^1_{q^1 v} \geq$*

$\Pi^0_{q^0v} \geq 0$ *and* $N\Pi^1_{q^1q^0} \geq n\Pi^0_{q^0q^1}$ *or* $\Pi^1_{q^1v} \geq \Pi^0_{q^0v}$, $\Pi^0_{q^0v} \leq 0$ *and* $N\Pi^1_{q^1q^0} \leq n\Pi^0_{q^0q^1}$, *then* $q^1_v - q^0_v \geq 0$.

*Proof of Corollary 1.* To save on notation,let's denote $\Pi^j_{q^jq^h}$ by $\Pi^j_{jh}$ and $\Pi^j_{q^jv}$ by $\Pi^j_{jv}$ for $j, h \in \{0, 1\}$. Then, totally differentiating the FOC for private and public firms concerning $v$ and imposing symmetry, we obtain that

$$q^1_v(v, g) = -\frac{\Pi^1_{1v}(\Pi^0_{00} + (N-1)\Pi^0_{0N}) - N\Pi^0_{0v}\Pi^1_{10}}{(\Pi^0_{00} + (N-1)\Pi^0_{0N})(\Pi^1_{11} + (n-1)\Pi^1_{1n}) - nN\Pi^0_{01}\Pi^1_{10}}.$$

Because the $B_0$-matrix assumption implies that the denominator is positive, a sufficient condition for $q^1_v(v, g) > 0$ is the following: $\Pi^1_{1v}(\Pi^0_{00} + (N-1)\Pi^0_{0n}) - N\Pi^0_{0v}\Pi^1_{10} \leq 0$. A sufficient condition for this is that $\Pi^1_{1v} > 0$ and $\operatorname{sign}\Pi^0_{0v} = \operatorname{sign}\Pi^1_{10}$.

Also, we obtain that

$$q^0_v(v, g) = \frac{-n\Pi^0_{01}q^1_v(v, g) - \Pi^0_{0v}}{\Pi^0_{00} + (N-1)\Pi^0_{0N}}.$$

If $q^1_v(v, g) \geq 0$, the public-providers quality decreases with $v$ whenever $\Pi^0_{0v}$ and $\Pi^0_{01}$ are both non-positive. Substituting into for $q^1_v(v, g)$, $q^0_v(v, g) \leq 0$ whenever $n\Pi^1_{1v}\Pi^0_{01} - \Pi^0_{0v}(\Pi^1_{11} + (n-1)\Pi^1_{1n}) \geq 0$.

It follows from this that

$$q^1_v(v, g) - q^0_v(v, g) = q^1_v(v, g)\frac{\Pi^0_{00} + (N-1)\Pi^0_{0N} + n\Pi^0_{01}}{\Pi^0_{00} + (N-1)\Pi^0_{0N}} + \frac{\Pi^0_{0v}}{\Pi^0_{00} + (N-1)\Pi^0_{0N}}.$$

Substituting into for $q^1_v(v, g)$, $q^1_v(v, g) - q^0_v(v, g) \geq 0$ whenever

$$-\Pi^1_{1v}(\Pi^1_{11} + (n-1)\Pi^1_{1n} + n\Pi^0_{01}) + \Pi^0_{0v}(\Pi^1_{11} + (n-1)\Pi^1_{1n} + N\Pi^1_{10}) \geq 0.$$

Because $H^T(q; \beta)$ is a $B_0$ matrix, then this holds whenever $\Pi^1_{1v} \geq 0$ and $\Pi^0_{0v} \leq 0$ or $\Pi^1_{1v} \geq \Pi^0_{0v}$ and $N\Pi^1_{10} \geq n\Pi^0_{01}$.

□

To better understand the conditions under which a voucher can increase private and public providers' quality, we will split the analysis into two different cases: strategic complements and substitutes. We will provide the intuition in the case in which private providers' best response does not fall with the voucher.

**Strategic Complements** In this case, as private providers become more aggressive, public providers respond by being more aggressive, too. This feature describes the best of worlds for vouchers to have a positive impact on quality when an increase in the voucher raises private providers' quality. Qualities increase whenever private and public providers' best response raises (weakly) with the voucher; that is, $\Pi^1_{q^1v} \geq 0$ and $\Pi^0_{q^0v} \geq 0$ because the increase in $q^0$ increases the marginal return to $q^1$ and vice-versa. When either condition fails for public providers, both quality levels can increase when the complementarity is strong and $\Pi^1_{q^1v}$ is sufficiently large relative to the absolute value of $\Pi^0_{q^0v}$.

**Strategic Substitutes** When private providers increase quality, public providers want to offer a lower quality. For the voucher to be the tide that lifts all boats, the difference between $\Pi^1_{q^1v}$ and $\Pi^0_{q^0v}$ cannot be too large when $\Pi^0_{q^0v} > 0$, whereas there is no such tide when $\Pi^0_{q^0v} \leq 0$. In the former case, public providers' best responses increase similarly to private providers' best responses. This, together with the fact that the transpose of Hessian is $B_0$-matrix, ensures that the quality level offered by both providers increases. In the latter case, the tide that lifts all boats does not exist because private providers' best responses rise, which makes public providers less aggressive. Because a larger voucher makes them even less aggressive, their incentive to lower quality are stonger.

For private providers' best responses to increase with the voucher, the change in the sum of the business-stealing and strategic-commitment effect has to more than compensate for the increase in total marginal costs and the decrease in the profit margin when the pass-through from the voucher to prices is higher than $-1$.

A sufficient condition for the business-stealing effect to increase in the voucher is given by: i) the distribution of the best outside option that a consumer choosing private providers has; i.e., $G^n(\cdot)$ is convex. This implies that the intensity of competition is strong and holds when $n$ is sufficiently large irrespective of the shape of $g$; and ii) $\triangle U(q;\beta)$ is non-decreasing in $v$ for all $\theta$. This happens when $\theta \triangle q_v - \triangle p_v \geq 0$. This holds whenever $q_v^1(\theta - p_{q^1}^1 - p_{q^1}^0) \geq q_v^0(\theta - p_{q^0}^1 - p_{q^0}^0)$ .

## 5.4 Quality and Vouchers When Markets Are Mixed

We provide an answer to the following questions: i) Under what conditions does the introduction of private providers increase public providers' quality?; ii) Under what conditions privately-provided quality exceeds publicly-provided quality?; iii) When an increase in the voucher results in cream skimming; and iv) When does the introduction of private providers increase consumers' welfare?

Providing an answer to these questions relying on fundamentals only is not possible. We will have to focus on the symmetric-by-sector equilibrium and impose assumptions over equilibrium outcomes to answer them. From now on $(p^1, q^1)$ are the equilibrium price and quality of private providers and $(p^0, q^0)$ are the same of pubic providers.

Based on the results in Proposition 5 and Corollary 1, in this section, we assume the following

**Assumption 2.**

*i)* $q_v^1 - q_v^0 \geq 0.$

*ii)* $p_v^1 - p_v^0 \leq 0.$

Part (ii) demands counterweighing forces since

$$p_v^1 - p_v^0 = (p_v^1 - p_v^0)|_{q(v,g)=c} + (p_{q^1}^1 - p_{q^1}^0)q_v^1 + (p_{q^0}^1 - p_{q^0}^0)q_v^0 \leq 0.$$

This assumption ensures that $\triangle U(q(v,g);\theta)$ is non-decreasing with the voucher for any $\theta$ in any sub-game perfect equilibrium. This assumption biases the analysis towards favoring vouchers as having positive effects since they increase the benefit from choosing private providers over public providers without necessarily decreasing public-providers' utility. Thus, we provide vouchers with the best opportunity to have positive welfare effects.

This assumption establishes that an increase in vouchers toughens competition between public and private providers in the pricing and quality subgames.

### 5.4.1 Private Providers' Quality vs. Public Providers'Quality

Our following result provides sufficient conditions for private providers to offer a more significant quality than public providers.

**Proposition 9.** *Suppose within-sector-firms are symmetric, $(v, g) \in R^2_+$ is such that $p(q) > 0$, and $n = N$.*

*i) Suppose Assumption 2 part (ii) holds. If co-payments are allowed, there is a threshold $v^c(g, \beta^j)$ such that private providers' quality is larger than or equal to public providers' quality whenever $v \geq v^c(g, \beta)$.*

*ii) If co-payments are not allowed, there is a threshold $v^{nc}(g, \beta^j)$ such that private providers' quality is larger than or equal to public providers' quality whenever $v \geq v^{nc}(g, \beta)$.*

This shows that introducing free choice through private providers and vouchers induces higher privately-provided quality than publicly-provided quality when the voucher is sufficiently large.

On the one hand, an increase in the voucher, holding quality constant, lowers the price-cost margin since the pass-through from the voucher to prices is lower than -1 because prices are strategic complements, and private providers' best responses fall with the voucher. On the other hand, an increase in the voucher may decrease private providers' price elasticity of demand, which makes quality more profitable. This happens because a higher quality results in a business-stealing effect that results in ceteris paribus in customers switching from public to private providers, and the benefit of this is larger, the larger the price-cost margin, which rises when the price elasticity of demand is lower. Thus, a marginal increase in the voucher may or may not increase private providers' best response. However, because public providers' price-cost margin is independent of the voucher, the increase in the voucher does not partially compensate their drop in prices, and $p^1_v < p^0_v$, public providers' marginal quality benefit is smaller than that for private providers when $v$ is large.

### 5.4.2 Public Providers Quality Across Market Structures

The following result provides sufficient conditions for the existence private providers financed by vouchers to induce public providers to offer higher quality than they would do if they were to be the only providers in the market.

**Proposition 10.** *Suppose within-sector-firms are symmetric, $(v, g) \in R^2_+$ is such that $p(q) > 0$ and Assumption 2 holds.If $np^1_{q^0 v}(q(v, g)) + (N-1)p^0_{q^0 v}(q(v, g)) \leq 0$, $n \geq \dot{n}$, there is voucher threshold $v(\beta, g)$ such that $q^0(v, g) \geq q^0(g)$ for all $v \leq v(\beta, g)$.*

This shows when the quality difference $q^1 - q^0$ does not decrease with the voucher and the positive marginal impact of quality on demand decreases as the price increases, a mixed market results in large publicly provided quality than a market exclusively served by public providers whenever the voucher is not too large.

This result provides sufficient conditions for the voucher private providers to crowd-our public providers' quality.

The condition $n \geq \dot{n}$ is satisfied for all $n$ when $G^n(\cdot)$ is convex. This implies that the slope of the demand concerning its own price and the slope concerning its quality, holding prices, falls with $\triangle U$. Because this represents the distribution of the consumers' best public provider alternative to private provider $j$, the convexity of $G^n(\cdot)$ implies its density is increasing in $\epsilon$, which means that public providers are an attractive alternative to private providers. As mentioned above, convexity is always guaranteed for $n$ sufficiently large irrespective of the shape of $g$ since private providers have a small probability of winning customers over public providers, which means that private providers should be aggressive in providing customers with the highest utility.

### 5.4.3 Cream-Skimming

Here, we find sufficient conditions for vouchers to result in cream skimming; customers who value the quality the most self-select into the private sector.

Let's define $\triangle U(q, \theta) \equiv U(y, q^1, p^1(q), \theta) - U(y, q^0, p^0(q), \theta)$. Then, the average quality valuation in the private sector conditional on the private sector being chosen is given by

$$\theta^1(v) = \frac{\mathbb{E}_{\epsilon,\theta}[\theta G^{n-1}(\epsilon) G^N\big(\triangle U(q;\theta)+\epsilon\big)]}{\mathbb{E}_{\epsilon,\theta}[G^{n-1}(\epsilon) G^N\big(\triangle U(q;\theta)+\epsilon\big)]}$$

and the average quality valuation in the public sector, conditional on the public sector being chosen, is

$$\theta^0(v) = \frac{\mathbb{E}_{\epsilon,\theta}[\theta G^{N-1}(\epsilon) G^n\big(-\triangle U(q;\theta)+\epsilon\big)]}{\mathbb{E}_{\epsilon,\theta}[G^{N-1}(\epsilon) G^n\big(-\triangle U(q;\theta)+\epsilon\big)]}.$$

**Proposition 11.** *Suppose within-sector-firms are symmetric, $(v,g) \in R^2_+$ is such that $p(q) >> 0$ and Assumption 2. If $q^1(v,g) \geq q^0(v,g)$, then an increase in the voucher increases cream-skimming in the private sector.*

For an increase in the voucher to result in cream-skimming, several conditions must be satisfied. The most important ones are that the difference between private providers and public providers' marginal profit from quality does not decrease with the voucher and privately-provided quality exceeds publicly-provided quality. This assumptions ensure that $\bar{\theta}^1(v)$ is TP-2 in $(q^1, v)$. The first part implies that the difference between the non-random component of the utility for private providers and that for public providers does not decrease with the voucher. The positive difference in qualities is necessary for the same difference to increase with quality valuation $\theta$.

### 5.4.4 Costumer Welfare

This section compares consumer welfare before and after private providers and vouchers are introduced in the market. We provide sufficient conditions under which the introduction of free choice is welfare-enhancing. A direct benefit for consumers of introducing free choice is that prices go down when holding quality constant, and there is room for improving the matching between heterogeneous families and firms. However, this may come with the cost of lower quality in either sector.

Customers' welfare is given by

$$\begin{aligned} W(v,g,\beta^j) \equiv \mathbb{E}_{\theta,\epsilon}\Big[& n\big(\bar{y}+\theta q^1 - p^1 + \epsilon\big) G^{n-1}(\epsilon) G^N\big(\triangle U(q,\theta)+\epsilon\big) + \\ & N\big(\bar{y}+\theta q^0 - p^0 + \epsilon\big) G^{N-1}(\epsilon) G^n\big(-\triangle U(q,\theta)+\epsilon\big)\Big]\Big|_{q=q(v,g)}. \end{aligned}$$

When there are $N_g$ public providers and no private providers, welfare is given by

$$W(g, \beta^j) \equiv N^g \left( \bar{y} + \bar{\theta} q^0 - p^0 + \mathbb{E}_\epsilon [\epsilon G^{N_g - 1}(\epsilon)) \right) \Big|_{q^0 = q^0(g)}.$$

**Proposition 12.** *Suppose within-sector-firms are symmetric,* $(v, g) \in R^2_+$ *is such that* $p(q) >> 0$*, Assumption 2, and there are* $N_g$ *public providers in market supplied only by them, n private providers and N public providers, with* $n + N = N_g$*, in a market supplied by both types of providers. If* $n \geq \dot{n}$*,* $p^1_v(v, g) \leq 0$*, there is a threshold* $v(g, \beta)$*, such that welfare increases whenever* $v \geq v(g, \beta)$*.*

# 6 Empirical Evidence: Education Markets

During the last two decades, school choice has been at the center of the debate in many countries that have enacted policies to improve academic achievement.

The empirical evidence regarding the impact of competition and vouchers on quality is mixed (see, for instance, **?**, **?**, and **?**). In general, competition and vouchers have an ambiguous effect on quality, and social stratification is crucial to understand the impact of different voucher policies on efficiency (quality) and equity. Overall, the evidence does not favor generalized vouchers but suggests that there are potential gains from adequately designed voucher policies.

The Chilean generalized voucher program is the largest-scale program in the world and is unique in that competition between for-profit voucher schools and public schools was in place between 1981 and 2015. By 2015, more than half of the pupils attend for-profit voucher schools, where costs are covered by both the state and parents, who pay on-average USD$400 per child a year. Around 8% attend to for-profit private schools, and the rest to free public schools. The evidence from Chile points out to a potent cream skimming and, at best, mixed evidence on the impact of vouchers on test scores (see, for instance, **?**, **?**, **?**, **?**). The mixed evidence on performance, the evidence on cream-skimming, and the one on the poor performance of public schools is consistent with our results when the voucher is higher than the corresponding thresholds derived in Propositions 9 and 11.

Furthermore, this is consistent with an increase in consumer welfare (Propostion 12).

Sweden also has a large-scale voucher program. Research finds that voucher schools have mixed results, and there are concerns related to grade inflation among voucher schools. The evidence shows that voucher competition has improved the performance of public schools and that the program design has contributed to limiting cream skimming. According to Proposition 10, introducing for-profit voucher schools results in larger public providers' quality when the voucher is lower than a given threshold.

In high-income countries, the impact of small-scale programs on test scores is sometimes adverse and sometimes favorable, but it is frequently the case that no significant effect is found. The most robust finding is that vouchers induce public schools to improve. In addition, recent evidence from small-scale experiments in the United States finds substantial gains in years of school for recipients who had not experienced gains in test scores. These results do not control for peer effects, yet the effects are significant by the standards of the peer effects literature and therefore encouraging concerning the impact of vouchers. There are positive reduced form findings from Colombia, although questions remain as to whether the central mechanisms that account for these are due to vouchers. In India, vouchers result in minor improvements in test scores, they achieve that at one-third the cost per student of public schools and with no adverse distributional effects.

In the United States, charter schools are privately operated but publicly funded and tuition-free, which provide the possibility of school choice. An extensive literature based on lottery-based designs that account for student selection establishes that charter schools can improve student learning and later-life outcomes. This literature has focused on oversubscribed charters—often located in urban areas—as this is necessary for the lottery design. Angrist, Pathak, and Walters (2013) and Place and Gleason (2019) find that charter schools in nonurban areas do not improve student achievement and suggest substantial heterogeneity in the effect of attending a charter school. See Chabrier, Cohodes, and Oreopoulos (2016) for a more detailed, up-to-date review and contextualization of the results from charter school lotteries.

Evidence from various contexts indicates that parents and students view schools as differentiated products and select schools based on idiosyncratic match (Hastings, Kane, and Staiger 2006; Walters 2018).

The evidence on district-wide school choice plans shows that being admitted to a preferred school yields minor test score impacts. For instance, **?** study, using randomized lotteries to evaluate the Louisiana Scholarship Program, a voucher plan that provides public funds for disadvantaged students to attend private schools, finds that participation lowers math scores by 0.4 standard deviations and also reduces achievement in reading, science, and social studies.[18] The effect of non-urban charter schools on achievement is mixed; urban charter schools show positive causal effects on achievement, and studies based on admission lotteries regarding charters in Boston and New York show that they substantially improve academic achievement. **?** finds that tastes for charter schools among Boston students are negatively associated with achievement gains. This means that low-achievers, poor students, and those with weak unobserved tastes for charters gain the most from charter attendance but are less likely to apply. This, together with the positive effect on achievement, suggests that charter school choices are inconsistent with sorting based on comparative advantage in academic achievement and a greater willingness of motivated parents to seek out effective schools and invest more in human capital on other dimensions.

This stems from the fact that only those with strong preferences for private schools would choose them because they offer lower quality. If a voucher is imposed when this is the equilibrium, then the resulting outcome could explain the case of the Louisiana Scholarship Program or the finding about Boston's charter schools. In the case in which the sub-game perfect equilibria predicts private schools with vouchers competing against the low-quality existent public schools, public schools' quality should be smaller than that of private schools, and therefore, selection will be driven more by quality differences than strong preferences for voucher schools. Hence, the school system will be segregated more on quality than on other dimensions like religious or vocational education. This feature has the potential to explain the documented cream-skimming of the Chilean generalized voucher system. In both cases, the differences in quality between the public and private

[18] These effects may be partly due to the selection of low-quality private schools into the program. Thus, peer preferences weaken the argument in favor of free choice.

sectors should be more minor than the one predicted by the model since, in the first case, less is spent on publicly-provided quality and, in the second, less is spent on the voucher and more on publicly-provided quality.

# 7 Conclusions

This paper shows that when the goal of public providers is to maximize a linear combination of market share and profits, the free-to-choose rationale for quality is much more nuanced than suggested by his main proposer (**?**). Namely, the conditions for private competition partially financed by vouchers to increase quality are very stringent and rest on how competition and vouchers affect the elasticity of demand concerning both prices and quality when prices are chosen after qualities. Sufficient conditions are provided for aggregated quality, private quality, and cream skimming to rise with the voucher, and public quality and consumer welfare to rise when competition and vouchers are introduced. The analysis suggests that policy makers must understand the market carefully before proposing to introduce private competition and vouchers as a solution to the low quality offered by public providers.

The analysis suggests the following avenues for future research. Firstly, quality competition should be studied when either prices are regulated and/or private providers are not-for-profits. For instance, co-payments could be fixed or a decreasing function of vouchers or subsidies that private providers receive. Secondly, to study targeted-by-income vouchers. This will prevent subsidizing relatively high-income customers who would patronize an expensive provider without a voucher. Thirdly, equilibria with incomplete coverage should be studied. In this case, vouchers not only lower prices but also increase coverage, affecting the selection of customers in terms of income across sectors.

# A Proofs of Results in Section 4

*Proof of Proposition 1.* Because $G^k$ are identically distributed, for all $j \in \mathcal{J}$ and $j \in \mathcal{J}^0$

$$D^j(p,q) = \mathbb{E}_{y,\theta,\epsilon}\Big[\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\backslash\{j\}} G^k\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Big]$$

and for all for all $j \in \mathcal{J}$,

$$\begin{aligned} D^j_j(p,q) = -\mathbb{E}_{y,\theta,\epsilon}\Big[\sum_{k\in\mathcal{J}\cup\mathcal{J}^0\backslash\{j\}} & \nu_g\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\times \\ & \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\backslash\{j\}} G\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Big] < 0 \end{aligned}$$

where $\nu_g(\cdot) \equiv g(\cdot)/G(\cdot)$,

$$\begin{aligned} D^j_k(p,q) = \mathbb{E}_{y,\theta,\epsilon}\Big[ & \nu_g\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\times \\ & \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\backslash\{j\}} G\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Big] > 0 \end{aligned}$$

$D^j(p,q)$ is strictly decreasing in $p^j$ and is strictly increasing in $p_{j'}$.

A log supermodular or MTP (multivariate totally positive of order 2) function is similarly defined for positive functions by $f(x \vee y)f(x \wedge y) > f(x)f(y)$. Thus. $D^j(p,q)$ is log-supermodular if $D^j(p \vee p',q)D^j(p \wedge p',q) > D^j(p,q)D^j(p',q)$. Because the multiplication of $TP_2$ functions is $TP_2$, the demand function is $TP_2$ if $G\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)$ is $TP_2$ in $(p^j,p_k)$. Observe that $G\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)$ can be written as $G\Big(K-(p^j-p^k)\Big)$. Let $p^{j'} \geq p^j$ and $p'_k \geq p_k$, this is $TP_2$ if and only if

$$\begin{aligned} & G\Big(K-(p^{j'}-p^{k'})\Big)G\Big(K-(p^j-p^k)\Big) \geq G\Big(K-(p^{j'}-p^k)\Big)G\Big(K-(p^j-p^{k'})\Big) \\ \Longleftrightarrow & G\Big(K-(p^{j'}-p_{k'})\Big)G\Big(K-(p^j-p^k)\Big) \geq G\Big(K-(p^{j'}-p_{k'})-z^k\Big)G\Big(K-(p^j-p^k)+z^k\Big), \end{aligned}$$

where $z^k = p^{k'} - p^k$. Observe that the RHS is decreasing in $z$ whenever

$$
\begin{aligned}
& -g\Big(K-(p^{j'}-p^{k'})-z_k\Big)G\Big(K-(p^j-p^k)+z_k\Big)+\\
& G\Big(K-(p^{j'}-p^{k'})-z^k\Big)g\Big(K-(p^j-p^k)+z^k\Big)\leq 0\\
& \Longleftrightarrow\\
& \nu_g\Big(K-(p^{j'}-p^{k'})+z^k\Big)\leq \nu_g\Big(K-(p^j-p^k)-z^k\Big).
\end{aligned}
$$

Because $G$ is log-concave and $z^k \geq 0$, this holds for all $z^k$. Because the inequality holds with equality when $z^k = 0$, the inequality holds for all $z^k > 0$. Thus, $D^j(p,q)$ is $TP_2$ in $p$. Because TP-2 is preserved under marginalization the demand is log-supermodular in $p$. We can proceed in the same way to show that is log-submodular in $q$ and log-supermodular in $(p^j, q^j)$.

Let $m = n + N$ and $d^m \in \Re^m$ be a vector of 1s and $b > 0$. Because demand depends on the difference in prices, we have that $D^j(p,q) = D^j(p + bd^m, q)$. Log-concavity in $p^j$ follows from this and the fact that $D^j(p,q)$ decreasing in $p^j$ and is $TP_2$. There is a well-known duality that a positive Lebesgue-measurable function, $f(x)$ on $\Re$, is log concave if and only if $f(x-y)$ is $TP_2$ in $x$ and $y$. Since monotone functions and continuous functions are Lebesgue-measurable, this duality holds for these functions.

Because $D^j(p,q)$ has increasing differences between $p^j$ and $p_{-j}$, for any $p_{jH} > p_{jL}$ and $b_H > b_L$, $D^j(p_{jH}, p_{-j} + b_H d^m, q) D^j(p_{jL}, p_{-j} + b_L d^m, q) \leq D^j(p_{jH}, p_{-j} + b_L d^m, q) D^j(p_{jL}, p_{-j} + b_H d^m, q)$. Because $D^j(p,q) = D^j(p + bd^m, q)$, we get that $D^j(p_{jH} - b_H, p_{-j}, q) D^j(p_{jL} - b_L, p_{-j}, q) \geq D^j(p_{jH} - b_L, p_{-j}, q) D^j(p_{jL} - b_H, p_{-j}, q)$. Hence, this implies that $D^j(p,q)$ is $TP_2$ in $p^j$ and $b$. Since $D^j(p,q)$ is decreasing in $p^j$, it is Lebesgue-measurable and therefore by **?**'s (**?**) result, $D^j(p,q)$ is log-concave by the duality between log concave functions and TP functions.

$$D^j_{q^j}(q)\big|_{p^j=k} = \mathbb{E}_{y,\theta,\epsilon}\Big[\theta \sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\times$$
$$\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Big] > 0,$$

for $k \in \mathcal{J}$

$$D^j_{q^k}(q)\big|_{p^j=k} = -\mathbb{E}_{y,\theta,\epsilon}\Big[\theta\nu_g\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\times$$
$$\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Big] < 0$$

Proceeding as before we can show that $D^j(p,q)$ is TP2 in $(p^j,q^j)$ and in $(p^j,-q^{-j})$.

Observe that for all $j,j' \in \mathcal{J}\cup\mathcal{J}^0$, is log-supermodular in $p$ if for each pair $p^j, p_{j'}$, the following holds

$$\frac{1}{D^j(p,q)} D^j_{j,j'}(p,q) - \frac{1}{(D^j(p,q))^2} D^j_j(p,q) D^j_{j'}(p,q) \geq 0.$$

Observe that

$$D^j_{jj}(p,q) = \mathbb{E}_{y,\theta,\epsilon}\Bigg[\Bigg(\sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu'_g\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)+$$
$$\Big(\sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Big)^2\Bigg)\times$$
$$\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\big)\Bigg],$$

$$
\begin{aligned}
D^j_{jk}(p,q) = -\mathbb{E}_{y,\theta,\epsilon}\Bigg[\Bigg( & \nu_g'\big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\big)+ \\
& \nu_g\big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\big)\times \\
& \sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Bigg)\times \\
& \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(U(y,q^j,p^j,\theta)+\epsilon^j - U(y,q^k,p^k,\theta)\Big)\Bigg],
\end{aligned}
$$

since $\nu_g' < 0$ because $g(\cdot)$ is log-concave.

□

*Proof of Proposition 2.* The proof follows closely the proof of Theorem 1 in **?**.

**Lemma 1.** *If $D(p,q)$ is strictly decreasing and log-concave in $p$, then for each $p_{-j}$,*

$$\Pi(p,q) \equiv (p^j + v - c(q^j))D(p,q),$$

*is continuous and strictly quasi-concave, and there is a unique $p$ that maximizes $\Pi(p,q)$.*

*Proof of Lemma 1.* In the proof of this lemma, we suppress $p_{-j}$ and the price subscripts and firms' superscripts for simplicity. Continuity is immediate, since convex functions and log-concave functions are continuous.

If $D(p,q)$ is strictly positive and log-concave in $p$, $D(p,q)$ is strictly convex in $p$ because

$$D(\lambda p_1 + (1-\lambda)p_2, q) > D(p_1,q)^{\lambda} D(p_2,q)^{1-\lambda},$$

for $D(p_1,q) \neq D(p_2,q)$ and $\lambda \in (0,1)$.

Since $D(p,q)$ is strictly decreasing and convex, it has a strictly increasing and concave inverse $k(x)$, where $pD(p) = \frac{k(x)}{x}$. Let $z = \frac{1}{x} = \frac{1}{D(p,q)}$, then

$$pD(p,q) = \frac{c(x)}{x} = zc\left(\frac{1}{z}\right),$$

so that $k\left(\frac{1}{z}\right)$ is the inverse demand function. Since $k(x)$ is strictly increasing and strictly concave, $(x+v)k\left(\frac{1}{x}\right)$ is strictly concave. Hence

$$\Pi(p,q) = pD(p,q) + (v - c(q))D(p,q) = (z + v - c(q))k\left(\frac{1}{z}\right)$$

is strictly concave as the sum of strictly concave functions is strictly concave.

Since $D(p,q)$ is strictly decreasing, if $\Pi$ is strictly concave in demand, it is strictly quasi-concave in price. Hence, a maximizer is unique if it exists.

Since $D(p,q)$ is strictly decreasing and log-concave,

$$\lim_{p\to\infty} D(p,q) = \lim_{p\to\infty} pD(p,q) = 0, \quad \text{so that} \quad \lim_{p\to\infty} \Pi(p,q) = 0.$$

Because $D(p,q)$ is strictly positive and strictly decreasing, $\Pi(\hat{p},q) > 0$ for a sufficiently large $p$ such that $p + v - c(q) >> 0$, so that we can take a $\hat{p}$ such that $\Pi(\hat{p},q) > \epsilon$.

Since $\lim_{p\to\infty} \Pi(p,q) = (p + v - c(q^j))D(p,q) = 0$, there is a $\bar{p}$ such that

$$\Pi(\hat{p},q) > \Pi(p,q)$$

for all $p > \bar{p}$.

Since $\Pi(0,q) > \Pi(p,q)$ for any $0 > p$, any maximizer of $\Pi(p,q)$ is in $[0,\bar{p}]$, and it exists since $\Pi(p,q)$ is continuous and $[0,\bar{p}]$ is compact. We can proceed exactly in the same way with $\Pi(p,q;\beta^j)$

□

Let $R^j(p)$ be provider $j$'s best response.

**Lemma 2.** *Suppose that $D(p^j, p_{-j}, q)$ is strictly positive, strictly decreasing in $p$, $C(\cdot)$ is increasing and convex, and $\ln D(p^j, p_{-j}, q)$ has increasing differences. Then if $D(p^j, p_{-j}, q)$ is increasing in $p$ or $C(\cdot)$ is linear, $R(p)$ is increasing.*

*Proof of Lemma 2.* Since any function on the real line is quasi-supermodular, it is sufficient to show that $D(p^j, p^{-j})$ has the single-crossing property in order to apply Milgrom and

Shannon's monotonicity theorem. If $D(p^j, p^{-j})$ does not have the single-crossing property, there exist $p^{-jH} \geq p^{-jL}$ and $p^{jH} > p^{jL}$ such that

$$(p^{jL} + v - c(q^j))D(p^{jL}, p^{-jH}, q) > (p^{jH} + v - c(q^j))D(p^{jH}, p^{-jH}, q) \tag{A.1}$$

and

$$(p^{jH} + v - c(q^j))D(p^{jH}, p^{-jL}, q) > (p^{jL} + v - c(q^j))D(p^{jL}, p^{-jL}, q) \tag{A.2}$$

By multiplying (A.2) by $D(p^{jL}, p^{-jL}, q) - D(p^{jH}, p^{-jL}, q) > 0$ and (A.2) by $D(p^{jL}, p^{-jH}, q) - D(p^{jH}, p^{-jH}), q) > 0$, and adding up, we obtain:

$$(p^{jH} - p^{jL})(D(p^{jL}, p^{-jH}, q)D(p^{jH}, p^{-jL}, q) - D(p^{jL}, p^{-jL}, q)D(p^{jH}, p^{-jH}, q)) > 0$$

The left-hand side is non-positive because $p^H > p^L$ and the log-supermodularity of demand implies that $D(p^{jH}, p^{-jH}, q)D(p^{jL}, p^{-jL}, q) \geq D(p^{jL}, p^{-jH}, q)D(p^{jH}, p^{-jL}, q)$. This contradicts the hypothesis that profits do not satisfy the single-crossing property.

□

We can replicate these proofs for public providers and show their best responses are increasing functions as for private providers.

Let $R^j(p^{-j})$ be private provider $j$'s best response function (it is unique) and $R^{j0}(p^{-j})$ be public provider $j$'s best response function (it is unique). Let $d^m$, with $m = n + N$, be a vector of 1s and $b > 0$. Because demands depend on the difference in prices, we have that $D^j(p) = D^j(p + bd^m)$. Then, for all $\beta^j \in [0, 1]$,

$$\begin{aligned}
&\big(\beta^j(R^j(p^{-j}) + v - c_j) + 1 - \beta^j\big)D^j(R^j(p^{-j}), p^{-j}) \\
>&\big(\beta^j((R^j(p^{-j} + bd^{m-1}) - b) + v - c_j)) + 1 - \beta^j\big)D^j(R^j(p^{-j} + bd^{m-1}) - b, p^{-j}), \text{ bc max} \\
=&\big(\beta^j((R^j(p^{-j} + bd^{m-1}) - b) + v - c_j) + 1 - \beta^j\big)D^j(R^j(p^{-j} + bd^{m-1}), p^{-j} + bd^{m-1}), \\
&\text{bc } D^j(p) = D^j(p + bd^m)
\end{aligned}$$

and

$$
\begin{aligned}
&\big(\beta^j(R^j(p^{-j}+bd^{m-1})+v-c_j)+1-\beta^j\big)D^j(R^j(p^{-j}+bd^{m-1}),p^{-j}+bd^{m-1})\\
>&\big(\beta^j(R^j(p^{-j})+b)+v-c_j)+1-\beta^j\big)D^j(R^j(p^{-j}+bd^{m-1}),p^{-j}+bd^{m-1}),\ \text{bc}\max\\
=&\big(\beta^j(R^j(p^{-j})+b)+v-c_j)+1-\beta^j\big)D^j(R^j(p^{-j}),p^{-j}),\ \text{bc}\ D^j(p)=D^j(p+bd^m)
\end{aligned}
$$

We deduce from these two inequalities that

$$
\begin{aligned}
0>&b(D^j(R^j(p^{-j}+bd^{m-1}),p^{-j}+bd^{m-1})-D^j(R^j(p^{-j}),p^{-j}))\\
=&b(D^j(R^j(p^{-j}+bd^{m-1})-b,p^{-j})-D^j(R^j(p^{-j}),p^{-j})).
\end{aligned}
$$

Because demand is strictly decreasing, this implies that the best-response is single valued for each $j$ and $R^j(p^{-j}+bd^{m-1})<R^j(p^{-j})+b$ for all $b>0$.

Next, let's assume that there are two fixed points, denoted by $x$ and $y$. Let $e\equiv\max_{j\in\{1,\ldots,m\}}|x^j-y^j|$. Hence, $R(p)$ has two different fixed points. Observe that $x^{-j}\wedge y^{-j}\leq y^{-j}$, $x^{-j}\vee y^{-j}\geq y^{-j}$ and $x^{-j}\vee y^{-j}\leq x^{-j}\wedge y^{-j}+ed^{m-1}$. Because $R^j(x^{-j}+bd^{m-1})<R^j(x^{-j})+e$ and $R^j$ is increasing $|R^j(x^{-j})-R^j(y^{-j})|\leq(R^j(x^{-j}\wedge y^{-j}+bd^{m-1})-R^j(x^{-j}\wedge y^{-j}))<e$. This contradicts the fact that $|R^j(x)-R^j(y)|=|x-y|=e$. □

*Proof of Proposition 4.* Take any matrix $A$ which entries $a_{ij}$. Then matrix $A$ is $B_0$-matrix if and only if for all $i\in\mathcal{J}$, we have that

$$
\sum_{j=1}^{n}a_{ij}\geq n\max\Big\{0,a_{ij}|j\neq i\Big\}.
$$

Let $b$ be a matrix with elements $b_{ij}$ and matrix $A(b)_{jk}$ be the matrix resulting from substituting column $j$ per vector $b^k$. Then **?** shows that $(A(b)_{jk})^T$ is a B-matrix if vector $b$ satisfies the following

$$
\sum_{i=1}^{n}b_{ik}\geq n\max\Big\{0,b_{ik}|i\neq k\Big\},
$$

Let $y$ be a matrix with elements $y_{ij}$. Thus, if we have the system of equations $Ay=b$, using the Cramer's rule, we can show that

$$y_{jk} = \frac{\det\left(A(b)_{jk}\right)}{\det\left(A\right)},$$

and therefore $y_{jk} \geq 0$ if $A$ and $A(b)_{jk}$ are $B_0$-matrices. Theorem 1 in **?** shows that $y_i > 0$ whenever $A^T$ is a $B_0$ matrix and $b$ is mean positive in $i$.

It is easy to check that $-H$ is a $B_0$-matrix since $H$ is diagonally dominant and in each row, the off-diagonal elements of $-H$ are all negative and smaller than the diagonal element. Similarly, we can check that $(-H)^T$ is a B-matrix. Let $-H^{i,j}$ the co-factor $ij$ from matrix $-H$. Hence, $\sum_{j=1}^{0}(-H^{i,j}) \geq 0$ and $\det\left(-H\right) > 0$.

To see that $-\Pi$ is a B-matrix, observe that this requires that for all $i \in \mathcal{J}$,

$$-\sum_{j \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi_i(p, q; \beta^j)}{\partial p_i \partial p^j} \geq n \max_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \left\{0, -\frac{\partial^2 \log \Pi_i(p, q)}{\partial p_i \partial p^j} | j \neq i\right\},$$

which follows from the fact that

$$-\frac{\partial^2 \log \Pi_i(p, q; \beta^j)}{\partial p_i^2} > \sum_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \frac{\partial \log \Pi_i(p, q)}{\partial p_i \partial p^j},$$

and

$$-\frac{\partial^2 \log \Pi_i(p, q; \beta^j)}{\partial p_i \partial p^j} < 0, \ \forall j \neq i.$$

To see that $-H^T$ is a B-matrix, observe that this requires that for all $i \in \mathcal{J}$,

$$-\sum_{j \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi^j(p, q; \beta^j)}{\partial p^j \partial p_i} \geq n \max_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \left\{0, -\frac{\partial^2 \log \Pi^j(p, q; \beta^j)}{\partial p^j \partial p_i} | j \neq i\right\},$$

which follows from the fact that

$$-\frac{\partial^2 \log \Pi_i(p, q; \beta^j)}{\partial p_i^2} > \sum_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \frac{\partial \log \Pi^j(p, q; \beta^j)}{\partial p^j \partial p_i},$$

and

$$-\frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial p_i} < 0,\ \forall j \neq i.$$

Let's also define the matrix $-H(p,q;\beta^j)$ as the matrix with entries $\left\{\frac{\partial^2 \log \Pi_i(p,q;\beta^j)}{\partial p_i \partial q^j}\right\}_{i,j\in\mathcal{J}}$. Using Cramer's rule, we can show that

$$\frac{\partial p^j(q)}{\partial q^k} = \frac{\det\left(-H^{jk}(p,q;\beta)\right)}{\det\left(-H(p,q;\beta)\right)},$$

where $-H^{jk}(p,q;\beta)$ is the matrix obtained from $-H$ by replacing column $j$ with the column vector $k$ from $H(p,q;\beta)$.

Then, $-H^{jk}(p,q;\beta)$ is a B-matrix if and only if

$$\sum_{j\in\mathcal{J}\cup\mathcal{J}^0} \frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial q^k} \geq n \max\left\{0, \frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial q^k} | j \neq k\right\}, \tag{A.3}$$

If this holds $\det -H(p,q;\beta) > 0$.

Let's assume that $m = n + N$, then from the equilibrium conditions we deduce the following

$$H_{m,m}(p) \equiv \begin{pmatrix} \Pi^1_{1,1} & \Pi^1_{1,2} & \cdots & \cdots & \cdots & \cdots & \Pi^1_{1,m} \\ \Pi^2_{2,1} & \Pi^2_{2,2} & \cdots & \cdots & \cdots & \cdots & \Pi^2_{2,m} \\ \vdots & \cdots & \ddots & \cdots & \cdots & \cdots & \vdots \\ \Pi^k_{k,1} & \ddots & \cdots & \Pi^k_{k,k} & \cdots & \cdots & \Pi^k_{k,m} \\ \\ \Pi^{k+1}_{k+1,1} & \ddots & \cdots & \cdots & \Pi^{k+1}_{k+1,k+1} & \cdots & \Pi^{k+1}_{k+1,m} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \Pi^m_{m,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \Pi^m_{m,m} \end{pmatrix}$$

$$p_{q^j} \equiv \begin{pmatrix} p^1_{q^j} \\ \vdots \\ \vdots \\ p^n_{q^j} \\ \vdots \\ \vdots \\ p^m_{q^j} \end{pmatrix} \qquad b(q^j) \equiv \begin{pmatrix} -\Pi^1_{1,q^j} \\ \vdots \\ \vdots \\ -\Pi^n_{n,q^j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$H_{m,m}(p)p_{q^j} = b(q^j) \tag{A.4}$$

It follows then

$$p^i_{q^j} = -\frac{\sum_{i\in\mathcal{J}\cup\mathcal{J}^0}\Pi^i_{i,q^j}H^{ij}(p,q;\beta)}{\sum_{i\in\mathcal{J}\cup\mathcal{J}^0}\Pi^i_{i,j}H^{ij}(p,q;\beta)},$$

where $H^{ij}(p,q;\beta)$ is the co-factor $ij$.

Hence, this is non-negative if and only if $-\sum_{i\in\mathcal{J}\cup\mathcal{J}^0}\Pi^i_{i,j}H^{ij}(p,q;\beta) \geq 0$, which is the case when $H(p,q;\beta)$ is a $B_0$-matrix. This follows from substituting the $i$ row by a row of $-1$ and then the determinant of this matrix is $-\sum_{i\in\mathcal{J}\cup\mathcal{J}^0}H^{ij}(p,q;\beta)$, which is positive because the new matrix with -1s in row 1 is a $B_0$ matrix since the row sum of -1s is lower than or equal to $n$ times the lowest between zero and the smallest off-diagonal row element, which is $-1$.

If the Jacobian of the equilibrium conditions and its transpose are both $B_0$-matrices as it is the case here, then its sum is negative definite (see, **?**) and, thereby, **?**'s (**?**) diagonally strict concavity property holds. Hence, the equilibrium will be unique.

□

*Proof of Proposition 5.* Let's define $\triangle U(p,q;\theta) \equiv U(y,q^1,p^1,\theta) - U(y,q^0,p^0,\theta)$. In this case the equilibrium price profile is the unique solution to the following system of equations

$$(p^1+v-c(q^1))\mathbb{E}_{y,\theta,\epsilon}\Big((n-1)\nu_g(\epsilon)+N\nu_g\big(\triangle U(p,q;\theta)+\epsilon\big)\Big)\times$$
$$G^{n-1}(\epsilon)G^N\big(\triangle U(p,q;\theta)+\epsilon\big)=\mathbb{E}_{y,\theta,\epsilon}G^{n-1}(\epsilon)G^N\big(\triangle U(p,q;\theta)+\epsilon\big)$$

and

$$\big(\beta^j(p^0+g-c(q^0))+1-\beta^j\big)\mathbb{E}_{y,\theta,\epsilon}\Big((N-1)\nu_g(\epsilon)+n\nu_g\big(-\triangle U(p,q;\theta)+\epsilon\big)\Big)\times$$
$$G^{N-1}(\epsilon)G^n\Big(-\triangle U(p,q;\theta)+\epsilon\Big)=\beta^j\mathbb{E}_{y,\theta,\epsilon}G^{N-1}(\epsilon)G^n\big(-\triangle U(p,q;\theta)+\epsilon\big).$$

Let's define

$$M^1(p^1,p^0,q)\equiv-(p^1+v-c(q^1))\mathbb{E}_{y,\theta,\epsilon}\Big((n-1)\nu_g(\epsilon)+N\nu_g\big(\triangle U(p,q;\theta)+\epsilon\big)\Big)\times$$
$$G^{n-1}(\epsilon)G^N\big(\triangle U(p,q;\theta)+\epsilon\big)+\mathbb{E}_{y,\theta,\epsilon}G^{n-1}(\epsilon)G^N\big(\triangle U(p,q;\theta)+\epsilon\big)$$

and

$$M^0(p^1,p^0,q)\equiv-\big(\beta^j(p^0+g-c(q^0))+1-\beta^j\big)\mathbb{E}_{y,\theta,\epsilon}\Big((N-1)\nu_g(\epsilon)+n\nu_g\big(-\triangle U(p,q;\theta)+\epsilon\big)\Big)\times$$
$$G^{N-1}(\epsilon)G^n\Big(-\triangle U(p,q;\theta)+\epsilon\Big)+\beta^j\mathbb{E}_{y,\theta,\epsilon}G^{N-1}(\epsilon)G^n\big(-\triangle U(p,q;\theta)+\epsilon\big).$$

Observe that if $n=N$ and $g=v-\frac{1-\beta^j}{\beta^j}$, then at $q^1=q^0=\hat{q}$, $M^1(p^1(\hat{q},\hat{q}),p^0(\hat{q},\hat{q}),\hat{q},\hat{q})=M^0(p^1(\hat{q},\hat{q}),p^0(\hat{q},\hat{q}),\hat{q},\hat{q})=0$. Let's consider $q^1>q^0=\hat{q}$ and assume that the new prices are: $p^1(q^1,\hat{q})<p^1(\hat{q},\hat{q})$ and $p^0(q^1,\hat{q})>p^0(\hat{q},\hat{q})$.

$$\begin{aligned}0=&M^1(p^1(\hat{q},\hat{q}),p^0(\hat{q},\hat{q}),\hat{q},\hat{q})\\ \leq&M^1(p^1(q^1,\hat{q}),p^0(\hat{q},\hat{q}),\hat{q},\hat{q})\text{ by log}-\text{concavity in p}^1\\ \leq&M^1(p^1(q^1,\hat{q}),p^0(\hat{q},\hat{q}),q^1,\hat{q})\text{ by log}-\text{supermodularity in }(\text{p}^1,\text{q}^1)\\ \leq&M^1(p^1(q^1,\hat{q}),p^0(q^1,\hat{q}),\hat{q},\hat{q})\text{ by log}-\text{supermodularity in }(\text{p}^1,\text{p}^0)\end{aligned}$$

$$
\begin{aligned}
0 =& M^0(p^1(\hat{q},\hat{q}), p^0(\hat{q},\hat{q}), \hat{q}, \hat{q}) \\
\geq & M^0(p^1(\hat{q},\hat{q}), p^0(q^1,\hat{q}), \hat{q}, \hat{q}) \text{ by log}-\text{concavity in p}^1 \\
\geq & M^0(p^1(\hat{q},\hat{q}), p^0(q^1,\hat{q}), q^1, \hat{q}) \text{ by log}-\text{submodularity in }(\text{p}^0,\text{q}^1) \\
\geq & M^0(p^1(q^1,\hat{q}), p^0(q^1,\hat{q}), \hat{q}, \hat{q}) \text{ by log}-\text{supermodularity in }(\text{p}^1,\text{p}^0)
\end{aligned}
$$

We deduce from this that $q^1 > q^0 = \hat{q}$, $p^1(q^1,\hat{q}) < p^1(\hat{q},\hat{q})$, and $p^0(q^1,\hat{q}) > p^0(\hat{q},\hat{q})$ cannot be an equilibrium.

Totally differentiating the FOC for private and public firms and imposing symmetry, we obtain that

$$
p^1_v(q) = -\frac{\Pi^1_{1v}(\Pi^0_{00} + (N-1)\Pi^0_{0N})}{(\Pi^0_{00} + (N-1)\Pi^0_{0N})(\Pi^1_{11} + (n-1)\Pi^1_{1n}) - nN\Pi^0_{01}\Pi^1_{10}} < 0
$$

where the sign follows because $\Pi^1_{1v} < 0$ and the $B_0$-matrix assumption that implies that the numerator is negative and denominator positive. Also, we obtain that

$$
p^0_v(q) = p^1_v(q)\frac{-n\Pi^0_{01}}{\Pi^0_{00} + (N-1)\Pi^0_{0N}} < 0
$$

because $\Pi^0_{01} \geq 0$.

It follows from this that

$$
p^1_v(q) - p^0_v(q) = p^1_v(q)\frac{\Pi^1_{1v}(\Pi^0_{00} + (N-1)\Pi^0_{0N}) + n\Pi^0_{01}}{\Pi^0_{00} + (N-1)\Pi^0_{0N}} < 0,
$$

since the numerator is positive because $n\Pi^0_{01} \geq 0$, $\Pi^0_{00} + (N-1)\Pi^0_{0N}$ and the denominator are both negative because of the $B_0$-matrix property, and $p^1_v < 0$.

Totally differentiating the FOC for private and public firms with respect to $q^1$ and imposing symmetry, we obtain that

$$
p^1_{q^1}(q) = -\frac{\Pi^1_{1q^1}(\Pi^0_{00} + (N-1)\Pi^0_{0N}) - N\Pi^0_{0q^1}\Pi^1_{10}}{(\Pi^0_{00} + (N-1)\Pi^0_{0N})(\Pi^1_{11} + (n-1)\Pi^1_{1n}) - nN\Pi^0_{01}\Pi^1_{10}}
$$

where the sign follows because $\Pi^1_{1q^1} > 0$ and the $B_0$-matrix assumption that implies that the numerator is negative and denominator positive.

Also, we obtain that

$$p^0_{q^1}(q) = \frac{-n\Pi^0_{01} p^1_{q^1}(q) - \Pi^0_{0q^1}}{\Pi^0_{00} + (N-1)\Pi^0_{0N}}.$$

Substituting into for $p^1_{q^1}(q)$, this is negative whenever $n\Pi^1_{1q^1}\Pi^0_{01} \leq \Pi^0_{0q^1}(\Pi^1_{11} + (n-1)\Pi^1_{1n})$. If $\Pi^0_{0q^1} \geq 0$, this never holds, whereas if $\Pi^0_{0q^1} < 0$, we deduce the result from the inequality.

It follows from this that

$$p^1_{q^1}(q) - p^0_{q^1}(q) = p^1_{q^1}(q)\frac{\Pi^0_{00} + (N-1)\Pi^0_{0N} + n\Pi^0_{01}}{\Pi^0_{00} + (N-1)\Pi^0_{0N})} + \frac{\Pi^0_{0q^1}}{\Pi^0_{00} + (N-1)\Pi^0_{0N})}$$

Because of the $B_0$ property, a sufficient condition for this to hold is $\Pi^1_{1q^1} \geq 0$ and $\Pi^0_{0q^1} \leq 0$. □

*Proof of Proposition 9.* To find conditions for $q^1 \geq q^0$, let's evaluate public providers' first-order conditions in private providers' equilibrium quality and find conditions for which the marginal return to quality for private providers is larger than or equal to that for public providers. This requires the following to hold

$$-\frac{D^1(q)}{D^j_1(q)} D^1_{q^1}(q)|_{p^j=k} + \beta^j \frac{D^0(q)}{D^0_j(q)} D^0_{q^0}(q)|_{p^j=k} \geq -c^{j'}(q^1)(\beta^j D^0(q) - D^1(q)).$$

Let $\triangle(p) \equiv p^1 - p^0$. Because demands are independent of $q$ when $q^0 = q^1$, this re-writes as follows

$$(\bar{\theta}-c'(q^1))\Big(\mathbb{E}_\epsilon\Big[G^{n-1}(\epsilon)G^n(\epsilon-\triangle(p))\Big]-\beta^j\mathbb{E}_\epsilon\Big[G^{n-1}(\epsilon)G^n(\epsilon+\triangle(p))\Big]\Big)-$$
$$(n-1)\frac{D_k^1(p(q),q)}{D_j^1(p(q),q)}p^1_{q^1}(q)-n\frac{D_h^1(p(q),q)}{D_j^1(p(q),q)}p^0_{q^1}(q)+$$
$$\beta^j(n-1)\frac{D_h^0(p(q),q)}{D_j^0(p(q),q)}p^0_{q^0}(q)+\beta^j n\frac{D_k^0(p(q),q)}{D_j^0(p(q),q)}p^1_{q^0}(q)\geq 0,$$

where

$$D_j^1(p,q)=-\mathbb{E}_{y,\theta,\epsilon}\Big[\Big((n-1)\nu_g(\epsilon)+n\nu_g(\epsilon-\triangle(p))\Big)G^{n-1}(\epsilon)G^n(\epsilon-\triangle(p))\Big]<0,$$

,

$$D_k^1(p,q)=\mathbb{E}_\epsilon\Big[\nu_g(\epsilon)G^{n-1}(\epsilon)G^n(\epsilon-\triangle(p))\Big]>0,$$

$$D_h^1(p,q)=\mathbb{E}_\epsilon\Big[\nu_g(\epsilon-\triangle(p))G^{n-1}(\epsilon)G^n(\epsilon-\triangle(p))\Big]>0,$$

$$D_j^0(p,q)=-\mathbb{E}_\epsilon\Big[\Big((n-1)\nu_g(\epsilon)+n\nu_g(\epsilon+\triangle(p))\Big)G^{n-1}(\epsilon)G^n(\epsilon+\triangle(p))\Big]<0,$$

,

$$D_h^0(p,q)=\mathbb{E}_\epsilon\Big[\nu_g(\epsilon)G^{n-1}(\epsilon)G^n(\epsilon+\triangle(p))\Big]>0,$$

$$D_k^0(p,q)=\mathbb{E}_\epsilon\Big[\nu_g(\epsilon+\triangle(p))G^{n-1}(\epsilon)G^n(\epsilon+\triangle(p))\Big]>0.$$

Becasue $D^j$ is log-concave in $p^j$ and log-supermodular in $p$, then $D_k^j/D_j^j$ rises with $p^j$.

Observe that for all $k \in \mathcal{J} \cup \mathcal{J}^0$,

$$\begin{aligned}
&\text{sign}\left\{\frac{\partial(D^j_k/D^j_j)}{\partial p^1}\right\} = \text{sign}\left\{D^j_{kj}D^j_j - D^j_k D^j_{jj}\right\} \\
\geq &\text{sign}\left\{(D^j)^{-1}D^j_k(D^j_j)^2 - D^j_k D^j_{jj}\right\} = D^j_k\,\text{sign}\left\{(D^j)^{-1}(D^j_j)^2 - D^j_{jj}\right\} \\
\geq &0
\end{aligned}$$

where the first inequality follows from log-spm and the second from log-concavity.

Let's define

$$\alpha(\triangle(p)) \equiv \frac{(n-1)\mathbb{E}_\epsilon\left[\nu_g(\epsilon)G^{n-1}(\epsilon)G^n(\epsilon - \triangle(p))\right]}{\mathbb{E}_\epsilon\left[\left((n-1)\nu_g(\epsilon) + n\nu_g(\epsilon - \triangle(p))\right)G^{n-1}(\epsilon)G^n(\epsilon - \triangle(p))\right]} \in (0,1)$$

and

$$\gamma(\triangle(p)) \equiv \frac{(n-1)\mathbb{E}_\epsilon\left[\nu_g(\epsilon)G^{n-1}(\epsilon)G^n(\epsilon + \triangle(p))\right]}{\mathbb{E}_\epsilon\left[\left((n-1)\nu_g(\epsilon) + n\nu_g(\epsilon + \triangle(p))\right)G^{n-1}(\epsilon)G^n(\epsilon + \triangle(p))\right]} \in (0,1).$$

Then,

$$\begin{aligned}
&(\bar{\theta} - c'(q^1))\left(\mathbb{E}_\epsilon\left[G^{n-1}(\epsilon)G^n(\epsilon - \triangle(p))\right]\beta^j\mathbb{E}_\epsilon\left[G^{n-1}(\epsilon)G^n(\epsilon + \triangle(p))\right]\right) + \\
&\alpha(\triangle(p))p^1_{q^1}(q) + (1-\alpha(\triangle(p)))p^0_{q^1}(q) - \beta^j\gamma(\triangle(p))p^0_{q^0}(q) - \beta^j(1-\gamma(\triangle(p)))p^1_{q^0}(q) \geq 0.
\end{aligned}$$

Observe that if $\triangle(p) > 0$, then $\alpha(\triangle(p)) < \gamma(\triangle(p))$ since $\triangle(p)$ rises $p^1$, $\alpha(\triangle(p))$ falls with $\triangle(p)$ and $\gamma(\triangle(p))$ rises with $\triangle(p)$, and $\alpha(0) = \gamma(0)$. Because $p^1_{q^1}(q) \geq p^0_{q^1}(q)$ and $p^0_{q^0}(q) \geq p^1_{q^0}(q)$, the LHS falls with $\triangle(p)$. Because the inequality holds strictly for all $\triangle(p) \leq 0$ for all $\beta^j < 1$, by the Intermediate Value theorem there is a unique threshold $\widetilde{\triangle}$ such that the inequality holds for all $\triangle(p) \leq \widetilde{\triangle}$.

Because $p^1(q) - p^0(q)$ when $(p^1(q), p^0(q))$ are strictly positive is independent of $q$ when $q^1 = q^0$ and $p^1(q) - p^0(q)$ falls with $v$ since $|\Pi_{p^0p^0}(p,q;\beta^j)| \geq \Pi_{p^0p^1}(p,q;\beta^j)$, there is a threshold $v(g,\beta^j)$ such that $\triangle(p) \leq \widetilde{\triangle}$ for all $v \geq v(g,\beta^j)$.

When co-payments are not allowed, the result follows immediately from the fact that $q^1$ rises with $v$ whereas $q^0$ falls with $v$ since qualities are strategic substitutes, private providers' best responses raises with $v$, and public providers' best response is independent of $v$. $\square$

*Proof of Proposition 10.* To find conditions for $q^0(v,g) \geq q^0(g)$, let's evaluate publlic providers' first-order conditions when the market is supplied by public and private providers at the equilibrium quality for private providers and the equilibrium quality for public providers that emerge when the market is supplied only by public providers and find conditions under which public providers' marginal profits for quality in a mixed market are larger than or equal to that in a market supplied exclusively by public providers. This requires the following to hold

$$\Bigg(\frac{\mathbb{E}_{\theta,\epsilon}\Big[G^{N-1}(\epsilon)G^{n}(\epsilon-\triangle U(q;\theta))\Big]}{\mathbb{E}_{\theta,\epsilon}\Big[\Big((N-1)\nu_g(\epsilon)+n\nu_g(\epsilon-\triangle U(q;\theta))\Big)G^{N-1}(\epsilon)G^{n}(\epsilon-\triangle U(q;\theta))\Big]}\times$$
$$\Bigg(\mathbb{E}_{\theta,\epsilon}\Big[\theta\Big((N-1)\nu_g(\epsilon)+n\nu_g(\epsilon-\triangle U(q;\theta))\Big)G^{N-1}(\epsilon)G^{n}(\epsilon-\triangle U(q;\theta))\Big]+$$
$$np^1_{q^0}+(N-1)p^0_{q^0}\Bigg)\Big|_{q=(q^1(v,g),q^0(g))}-D^0(q^1(v,g),q^0(g))c'(q^0(g))-$$
$$\frac{\mathbb{E}_{\theta,\epsilon}\Big[G^{N_g-1}(\epsilon)\Big]}{\mathbb{E}_{\theta,\epsilon}\Big[(N_g-1)\nu_g(\epsilon)G^{N_g-1}(\epsilon)\Big]}\Big(\mathbb{E}_{\theta,\epsilon}\Big[\theta(N_g-1)\nu_g(\epsilon)G^{N_g-1}(\epsilon)\Big]+$$
$$(N_g-1)c'(q^0(g))\Big)-\mathbb{E}_{\theta,\epsilon}\Big[G^{N-1}(\epsilon)G^{n}(\epsilon)\Big]c'(q^0(g))\geq 0.$$

The term outside the first parenthesis is $-D^0/D^0_{p^0}$. Let denote this by

$$r^0(v)=\frac{\eta_1(v)}{\eta_2(v)}$$

where $\eta_i(v)\equiv\mathbb{E}_\theta[\eta_i(\theta,v)]$ for $i\in\{1,2\}$,

$$\eta_1(\theta,v)\equiv\mathbb{E}_\epsilon\Big[G^{N-1}(\epsilon)G^{n}(\epsilon-\triangle U(q;\theta))\Big]f(\theta)$$

and

$$\eta_2(\theta,v)\equiv\mathbb{E}_\epsilon\Big[\Big((N-1)\nu_g(\epsilon)+n\nu_g(\epsilon-\triangle U(q;\theta))\Big)G^{N-1}(\epsilon)G^n(\epsilon-\triangle U(q;\theta))\Big]f(\theta)$$

Then, $r^0(v)$ is monotonically increasing (decreasing) in $v\in\Re_+$ if and only if $\eta_i(v)$ is TP2 (RR2) in $(i,v)$, $(i,v)\in\{1,2\}\times\Re_+$; that is,

$$\eta_1(v_1)\eta_2(v_2)\geq\ (\leq)\eta_1(v_2)\eta_2(v_1) \tag{A.5}$$

with $v_1>v_2$.

Because $\eta_i(v)>0$ for $i\in\{1,2\}$, then $r^0(v)$ is increasing (decreasing) in $v$ if and only if for any $\lambda\in\Re$, for any $v\in\Re_+$,

$$\begin{aligned}\phi_\lambda(v)&=\int_{\theta\in\Theta}(\eta_1(\theta,v)-\lambda\eta_2(\theta,v))d\theta\\&=\int_{\theta\in\Theta}\mathbb{E}_\epsilon[G^{n-1}(\epsilon)G^N(\triangle U(q;\theta)+\epsilon)]\Big(1-\lambda\big((N-1)\nu_g(\epsilon)+n\nu_g(\epsilon-\triangle U(q;\theta))\big)\Big)dF(\theta)\end{aligned}$$

has at most one sign change and from negative (positive) to positive (negative) as $v$ increases on $\Re_+$ (see, **?**).

Because $\mathbb{E}_\epsilon[G^{n-1}(\epsilon)G^N(\triangle U(q;\theta)+\epsilon)]f(\theta)$ is positive for all $v$ when $\lambda\leq 0$, $\phi_\lambda(v)\geq 0$ for all $v$, whereas when $\lambda>0$, there is at most one sign change from positive to negative whenever $\triangle U(q;\theta)$ is non-increasing in $v$ which happens whenever $\theta\triangle q_v-\triangle p_v\geq 0$ for all $\theta$. Thus, $r^0(v)$ is non-increasing in $v$ whenever $\theta\triangle q_v-\triangle p_v\geq 0$ for all $\theta$.

Observe that the first term multiplying $-D^0/D^0_{p^0}$ is equal to $D^0_{q^0}(q)|_{p^1=c}$. This falls with $v$ whenever

$$\begin{aligned}&-\mathbb{E}_\epsilon\Big[(\theta\triangle q_v-\triangle p_v)\Big((N-1)\nu_g(\epsilon)\nu_g(\epsilon-\triangle U)+n\nu_g^2(\epsilon-\triangle U)+n\nu_g'(\epsilon-\triangle U)\Big)\times\\&\quad G^{N-1}(\epsilon)G^n(\epsilon-\triangle U)\Big]\leq 0.\end{aligned}$$

To sign this term, first we derive the next result.

**Claim 2.** *There is a threshold $\dot{n}$ such that $G^n(\cdot)$ is convex for all $n\geq n(v)$ irrespective of the shape of $g(\cdot)$.*

*Proof.* Let's define $m \equiv \min_{\epsilon\in[\underline{\epsilon},\bar{\epsilon}]} g(\epsilon)$ and $M \equiv \max_{\epsilon\in[\underline{\epsilon},\bar{\epsilon}]} |g'(\epsilon)|$. Because $g'$ is bounded on $[\underline{\epsilon},\bar{\epsilon}]$ and $M < \infty$. Then,

$$
\begin{aligned}
&n\nu_g^2(\epsilon-\triangle U)+\nu_g'(\epsilon-\triangle U)\\
=&\frac{1}{G^2(\epsilon-\triangle U)}\left((n-1)g^2(\epsilon-\triangle U)+G(\epsilon-\triangle U)g'(\epsilon-\triangle U)\right)\\
\geq&\frac{1}{G^2(\epsilon-\triangle U)}\left((n-1)m^2-M\right).
\end{aligned}
$$

For $n$ sufficiently large this is positive. □

This result, together with the fact that $(N-1)\nu_g(\epsilon) > 0$, implies that for all $n \geq n(v)$, $D^0_{q^0}(q)|_{p^1=c}$ falls with $v$ whenever $\theta\triangle q_v - \triangle p_v \geq 0$ for all $\theta$.

Next notice that

$$
\mathbb{E}_{\theta,\epsilon}\Big[G^{N-1}(\epsilon)G^n(\epsilon-\triangle U(q;\theta))\Big]\Big|_{q(v,g)} - \mathbb{E}_{\theta,\epsilon}\Big[G^{N-1}(\epsilon)G^n(\epsilon)\Big]
$$

falls with $v$ whenever

$$
\mathbb{E}_{\theta,\epsilon}\Big[(\theta\triangle q_v-\triangle p_v)G^{N-1}(\epsilon)(-1)G^n(\epsilon-\triangle U(q;\theta))v_g(\epsilon-\triangle U(q;\theta))\Big]\Big|_{q(v,g)} \leq 0.
$$

Let $\theta(q,p) \equiv \triangle p_v/\triangle q_v$. If $\triangle q_v \geq 0$, the first function has the single-crossing property In addition, if $\triangle q \leq 0$, the second function falls with $\theta$. Hence, because of the linearity of $\triangle U(q,\theta)$ in $\theta$, we have the following

$$
\begin{aligned}
&\mathbb{E}_{\theta,\epsilon}\Big[(\theta\triangle q_v-\triangle p_v)G^{N-1}(\epsilon)(-1)G^n(\epsilon-\triangle U(q;\theta))v_g(\epsilon-\triangle U(q;\theta))\Big]\\
\leq&(\bar{\theta}\triangle q_v-\triangle p_v)\mathbb{E}_{\epsilon}\Big[G^{N-1}(\epsilon)(-1)G^n(\epsilon-\triangle U(q;\theta(q,p)))v_g(\epsilon-\triangle U(q;\theta(q,p)))\Big]\\
&le0
\end{aligned}
$$

where the last inequality holds whenever $\bar{\theta}\triangle q_v - \triangle p_v \leq 0$.

Finally, let's assume that $np^1_{q^0v}(q) + (N-1)p^0_{q^0v}(q) \leq 0$. Then if $q^0(0,g) \geq q^0(g)$, by the Intermediate value theorem there is voucher threshold $v(\beta,g)$ such that $q^0(v,g) \geq$

$q^0(g)$ for all $v \leq v(\beta, g)$

It follows from this that if $n \geq n(v)$, $\triangle q_v \geq 0$, and $\triangle p_v \geq 0$, the function being integrated satisfies single-crossing property, and thereby $D^0_{q^0}(q)|_{p^1=c}$ is increasing in $v$.

When $H(p,q;\beta)$ and $H^T(q;\beta)$ are both $B_0$ matrices and $\Pi^1_{q^1 v}(p(q),q) \geq \Pi^0_{q^0 v}(p(q),q;\beta^j)$, $\triangle U$ raises with the voucher since $q^1_v - q^0_v \geq 0$ and $p^1_v - p^0_v \leq 0$. Because the first and third term are identical when $\triangle U = 0$, by the intermediate-value theorem, there is a threshold $\hat{v}$ such that the first-term is larger than the third one if $v \leq \hat{v}$.

Then, because $-D^0/D^0_{p^0}$ falls with $\triangle U$ and this rises with $v$, the LHS falls with the voucher $v$. Because $D^0_{\triangle U} = D^0_{p^0}$ and $\triangle U$ raises with $v$, the RHS falls with $v$. Thus, by the Intermediate-Value theorem there is a threshold $\dot{v}$ such that the inequality holds whenever $v \leq \dot{v}$. Hence, if the inequality holds at $v = 0$, there is a threshold $\tilde{v}$ such that inequality holds whenever $v \leq \tilde{v} \leq \min\{\dot{v}, \hat{v}\}$.

□

*Proof of Proposition 11.* Observe that $G^n(\cdot)$ is log-concave since $G(\cdot)$ is log-concave. Thus, $\int_\epsilon G^{n-1}(\epsilon) G^n(\triangle U + \epsilon) h(\theta) g(\epsilon) d\epsilon$ is log-concave whenever $\triangle q \geq 0$ since $h$ and $g$ are log-concave, the multiplication of log-concave functions is log-concave, and the integral of log-concave functions with constant limits is log-concave. The conditional mean is log-concave since both the denominator and the numerator are both log concave, $\theta$ is positive, and the domains for integration are convex sets.

**Definition** A real-valued function $f$ defined on an interval $I \subseteq \mathbb{R}$ is said to have at most one sign change from negative (positive) to positive (negative), as $x$ increases on $I$, if any one of the following three conditions is satisfied:

(i) $f(x) \geq 0$, $\forall x \in I$;

(ii) $f(x) \leq 0$, $\forall x \in I$;

(iii) If there exists $x_0 \in I$ such that $f(x_0) > 0$ (or $f(x_0) < 0$), then $f(x) > 0$ (or $f(x) < 0$), $\forall x \in [x_0, \infty) \cap I$.

The conditional mean can be written

$$\theta^1(v) = \frac{\eta_1(v)}{\eta_2(v)}$$

where $\eta_i(v) \equiv \mathbb{E}_\theta[\eta_i(\theta, v)]$ for $i \in \{1, 2\}$,

$$\eta_1(\theta, v) \equiv \mathbb{E}_\epsilon[\theta G^{n-1}(\epsilon) G^N(\triangle U(q;\theta) + \epsilon)] f(\theta)$$

and

$$\eta_2(\theta, v) \equiv \mathbb{E}_\epsilon[G^{n-1}(\epsilon) G^N(\triangle U(q;\theta) + \epsilon)] f(\theta)$$

Then, $\theta^1(v)$ is monotonically increasing (decreasing) in $v \in \Re_+$ if and only if $\eta_i(v)$ is TP2 (RR2) in $(i, v)$, $(i, v) \in \{1, 2\} \times \Re_+$; that is,

$$\eta_1(v_1)\eta_2(v_2) \geq \ (\leq)\eta_1(v_2)\eta_2(v_1) \tag{A.6}$$

with $v_1 > v_2$.

Because $\eta_i(v) > 0$ for $i \in \{1, 2\}$, then $\theta^1(v)$ is increasing (decreasing) in $v$ if and only if for any $\lambda \in \Re$, for any $v \in \Re_+$,

$$\begin{aligned}\phi_\lambda(v) &= \int_{\theta\in\Theta} (\eta_1(\theta, v) - \lambda\eta_2(\theta, v))d\theta \\ &= \int_{\theta\in\Theta} \mathbb{E}_\epsilon[G^{n-1}(\epsilon) G^N(\triangle U(q;\theta) + \epsilon)](\theta - \lambda) f(\theta) d\theta\end{aligned}$$

has at most one sign change and from negative (positive) to positive (negative) as $v$ increases on $\Re_+$.

Because $\mathbb{E}_\epsilon[G^{n-1}(\epsilon) G^N(\triangle U(q;\theta) + \epsilon)] f(\theta)$ is positive for all $v$ when $\lambda \leq 0$, $\phi_\lambda(v) \geq 0$ for all $v$, whereas when $\lambda > 0$, $\phi_\lambda(v)$ is either positive or negative for all $v$ and therefore it has no sign change and thereby it has at most one sign change from negative to positive (see, **?**).

□

*Proof of Proposition 12.* Welfare can be written as follows

$$
\begin{aligned}
W(v,g,\beta) =& \mathbb{E}_{\theta,\epsilon}\Big[n\triangle U(q,\theta)G^{n-1}(\epsilon)G^{N}\big(\triangle U(q,\theta)+\epsilon\big)\Big]+\\
& \mathbb{E}_{\theta,\epsilon}\Big[\Big(\bar{y}+\theta q^0-p^0+\epsilon\Big)\times\\
& \Big(NG^{N-1}\Big(\epsilon\Big)G^{n}\Big(-\triangle U(q,\theta)+\epsilon\Big)+nG^{n-1}\Big(\epsilon\Big)G^{N}\Big(\triangle U(q,\theta)+\epsilon\Big)\Big)\Big]\Big|_{q=q(v,g)}
\end{aligned}
$$

We deduce from this that consumer welfare rises when free choice is introduced whenever

$$
\begin{aligned}
&\Bigg(\mathbb{E}_{\theta,\epsilon}\Big[n\triangle U(q,\theta)G^{n-1}(\epsilon)G^{N}\big(\triangle U(q,\theta)+\epsilon\big)\Big]+\bar{y}-p^0+\mathbb{E}_{\theta,\epsilon}\Big[\Big(\theta q^0+\epsilon\Big)\times\\
&\Big(NG^{N-1}(\epsilon)G^{n}\big(-\triangle U(q,\theta)+\epsilon\big)+nG^{n-1}(\epsilon)G^{N}\big(\triangle U(q,\theta)+\epsilon\big)\Big)\Big]\Bigg)\Big|_{q(v,g)}\geq\\
&\Big(\bar{y}+\bar{\theta}q^0-p^0+N_g\mathbb{E}_{\epsilon}[\epsilon G^{N_g-1}(\epsilon)]\Big)\Big|_{q^0(g)}
\end{aligned}
$$

Observe that

$$
(\theta q^0+\epsilon)\big(NG^{N-1}(\epsilon)G^{n}\big(-\triangle U(q,\theta)+\epsilon\big)+nG^{n-1}(\epsilon)G^{N}\big(\triangle U(q,\theta)+\epsilon\big)\big) \tag{A.7}
$$

increases with $\theta$ whenever

$$
\begin{aligned}
&nN\triangle q\Big[\Big(G^{n-1}(\epsilon)G^{N}\big(\triangle U(q,\theta)+\epsilon\big)\nu_g\big(\triangle U(q,\theta)+\epsilon\big)-\\
&G^{N-1}(\epsilon)G^{n}\big(-\triangle U(q,\theta)+\epsilon\big)\nu_g\big(-\triangle U(q,\theta)+\epsilon\big)\Big)\Big]+\\
&q^0\big[\big(NG^{N-1}(\epsilon)G^{n}\big(-\triangle U(q,\theta)+\epsilon\big)+nG^{n-1}(\epsilon)G^{N}\big(\triangle U(q,\theta)+\epsilon\big)\big)\big]
\end{aligned}
$$

Let's define $\theta(v,g)\equiv\max\{\theta_L,\min\{\theta_H,\triangle p/\triangle q\}\}$ and observe that if $\theta(v,g)\in(\theta_L,\theta_H)$, which requires sign $\triangle q=$ sign $\triangle p$, then $\triangle U(q,\theta(v,g))=0$.

If $G^n(\cdot)\nu_g(\cdot)$ is non-decreasing, which holds whenever $G^n(\cdot)$ is convex, and $\triangle q\geq 0$, the first term is negative for all $\theta<\theta(v,g)$, zero at $\theta=\theta(v,g)$, and positive otherwise. Hence, the benefit from quality and non-pecuniary benefits in equation (A.7) is first decreasing and then increasing. Furthermore, observe that

$$\begin{aligned}&\mathbb{E}_{\theta,\epsilon}\big[(\theta(v,g)q^0+\epsilon)\big(NG^{N-1}(\epsilon)G^n\big(-\triangle U(q,\theta(v,g))+\epsilon\big)+\\&nG^{n-1}(\epsilon)G^N\big(\triangle U(q,\theta(v,g))+\epsilon\big)\big]\\=&\theta(v,g)q^0+N_g\mathbb{E}_\epsilon\big[\epsilon G^{N_g-1}(\epsilon)\big].\end{aligned}$$

Hence, when $\triangle q\geq 0$ and $G^n(\cdot)$ is convex, the LHS is larger than

$$\begin{aligned}&\mathbb{E}_{\theta,\epsilon}\big[(\theta q^0+\epsilon)\big(NG^{N-1}(\epsilon)G^n\big(-\triangle U(q,\theta)+\epsilon\big)+nG^{n-1}(\epsilon)G^N\big(\triangle U(q,\theta)+\epsilon\big)\big)\big]\\\geq&\theta(v,g)q^0+N_g\mathbb{E}_\epsilon\big[\epsilon G^{N_g-1}(\epsilon)\big].\end{aligned}$$

Next, notice that $\triangle U(q,\theta)$ is increasing in $\theta$ whenever $\triangle q>0$ negative for all $\theta\leq\theta(v,g)$ and positive otherwise. Thus, $\triangle U(q;\theta)$ has the single-crossing property in $\theta$. Also, notice that $G^{n-1}(\epsilon)G^N\big(\triangle U(q,\theta)+\epsilon\big)$ increases with $\theta$ whenever $\triangle q\geq 0$. Because $\triangle U(q,\theta)$ is linear in $\theta$, we have the following

$$\mathbb{E}_{\theta,\epsilon}\Big[n\triangle U(q,\theta)G^{n-1}(\epsilon)G^N\big(\triangle U(q,\theta)+\epsilon\big)\Big]\geq(\bar{\theta}\triangle q-\triangle p)\mathbb{E}_{\theta,\epsilon}\Big[nG^{n-1}(\epsilon)G^N(\epsilon)\Big].$$

We deduce from the discussion up to here that

$$\begin{aligned}W(v,g;\beta)\geq&(\bar{\theta}\triangle q-\triangle p)n\mathbb{E}_{\theta,\epsilon}\Big[G^{N_g-1}(\epsilon)\Big]+\\&\bar{y}-p^0(v,g)+\theta(v,g)q^0(v,g)+N_g\mathbb{E}_\epsilon\big[\epsilon G^{N_g-1}(\epsilon)\big]\end{aligned}$$

Hence, consumer's welfare under free choice is larger than under a market supplied only by public providers whenever

$$\frac{n}{N_g}(\bar{\theta}\triangle q-\triangle p)+\theta(v,g)q^0(v,g)-p^0(v,g)\geq\bar{\theta}q^0(g)-p^0(g).$$

If $n\geq\dot{n}$, $n+N=N_g$, $\triangle q\geq 0$, $q_v^1-q_v^0\geq 0$, $p_v^1\leq 0$ and $p_v^1-p_v^0\leq 0$, by the intermediate-value Theorem, there is a threshold $v(g,\beta)$ such that welfare under a mixed market exceeds that under an exclusively public market whenever $v \geq v(g,\beta)$.

$\square$